# Basics of OPTICS OF MULTILAYER SYSTEMS 

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## Contents

Preface .....  .4
Chapter 1SPECTRAL CHARACTERISTICS OF MULTILAYER COATINGS:THEORY5
1.1 Electromagnetic field in layered media; spectral characteristics of layered media ..... 5
1.1.1 Layered medium electromagnetic field equations .....  5
1.1.2 Amplitude transmittance and reflectance of the layered medium ..... 12
1.1.3 The Fresnel formulas ..... 18
1.1.4 Other ways of determining amplitude transmittance and reflectance of the layered medium ..... 19
1.1.5 Transmittance, reflectance and absorptance ..... 20
1.2 Principal recurrent methods for calculating layered media spectral coefficients ..... 22
1.2.1 Matrix calculating method ..... 22
1.2.2 Description of optical coating and presentation of calculation results. ..... 28
1.2.3 Recurrent formulas for the amplitude transmittance and reflectance. ..... 30
1.2.4 Approximate formulas. ..... 33
1.2.5 Other recurrent methods ..... 34
1.2.6 The admittance phase plane. ..... 38
1.3 Some common properties of multilayer coatings ..... 46
1.3.1 The relation between the characteristic matrix and spectral coefficients of a multilayer coating ..... 47
1.3.2 Relations between spectral characteristics for the direct and reverse waves ..... 51
1.3.3 Transmittance and reflectance of the combination of two multilayer subsystems ..... 54
1.4 The multilayer coatings optical properties: impact of variations in the layer parameters ..... 58
1.4.1 Spectral coefficients derivatives with respect to layer parameters ..... 58
1.4.2 Impact of the layer parameter errors on the multilayer coatings spectral coefficients: probability assessment method ..... 63
1.4.3 Multilayer coating spectral properties: influence of separate layer parameter variations ..... 66
1.5 Multilayer periodic systems. Quarter-wave dielectric mirrors ..... 68
1.5.1 Characteristic matrix of a periodic multilayer system ..... 69
1.5.2 Quarter-wave mirror properties at the central wavelength ..... 72
1.5.3 Width of the quarter-wave mirror high-reflectance zones ..... 76
1.5.4 Principal properties of a quarter-wave dielectric mirror in the proximity to the central wavelength ..... 80
1.5.5 Dielectric mirrors on a metal substrate ..... 84
1.6 Approximate formulas for dielectric narrow bandpass filters ..... 88
1.6.1 Common expressions for spectral coefficients of filter-type systems near central wavelength ..... 88
1.6.2 Research of narrow bandpass filters with a half-wave central layer ..... 92
Chapter 2
SYNTHESIS OF MULTILAYER OPTICAL COATINGS ..... 97
2.1 Synthesis methods based on merit function optimization ..... 98
2.1.1 Merit function selection ..... 98
2.1.2 Merit function optimization ..... 101
2.1.3 General pattern of the synthesis and the problem of the starting design choice ..... 108
2.2 Synthesis method based on the needle-like variations of the refractive index ..... 114
2.2.1 Needle-like variations of the refractive index ..... 115
2.2.2 The principal idea of the method ..... 119
2.2.3 Description of the method ..... 124
Bibliography ..... 130

## Preface

Multilayer optical coatings are successfully employed in various fields, such as optical and scientific instrumentation manufacturing, astronomy, spectroscopy, medicine, etc. A great number of laboratories and experts are involved in designing and manufacturing modern optical coatings. At present, in order to receive best results one needs to be aware of all aspects of design and production of multilayer optical coatings. The goal of this book is to provide a comprehensive outlook on modern problems in multilayer optics. A characteristic feature of the book is an attempt to make it useful for a wide range of experts from specialists of practical applications to specialists elaborating the theory of thin film coatings.

## Chapter 1

## SPECTRAL CHARACTERISTICS OF MULTILAYER COATINGS: THEORY

### 1.1 Electromagnetic field in layered media; spectral characteristics of layered media

It is essential to begin the section with deriving layered medium field equations from the Maxwell equations as sometimes errors occur in calculating due to an inconsistent use of some final formulas. It is evident from the following that a change of the sign in the time exponential factor results in the change of all values for complex conjugates. Misunderstanding is sometimes caused by the fact that different authors use formulas obtained with different choice of the time exponential factor. Also, introducing the reflectance through a magnetic or an electric field results in different signs in the final reflectance formulas. Knowledge of the principal steps in deriving formulas helps to eliminate possible error.

### 1.1.1 Layered medium electromagnetic field equations

Fig. 1.1 presents the basic physical model dealt with in studying multilayer optical coatings. Two homogeneous semi-infinite isotropic media are separated by a set of plane parallel isotropic layers, non-limited in either directions, whose permittivity and conductivity depend on one spatial coordinate perpendicular to the boundary between layers and media. A plane electromagnetic wave incites from the first medium to the boundary. This causes a plane reflected wave in the first medium and a plane transmitted wave in the second one.


Fig. 1.1. Multilayer optical coating model.

The given model adequately describes multilayer coatings as the width of the coating always exceeds many times the wavelength of the incident wave and the total thickness of the coating. Further, we consider these both media and the layers non-magnetic (magnetic permeability is equal to 1 ) and assume an absence of volume charges, both assumptions are quite standard in the multilayer optics. These assumptions admitted, the Maxwell equations in the medium are as follows:

$$
\begin{equation*}
\nabla \times \vec{E}=-\frac{1}{c} \frac{\partial \vec{H}}{\partial t}, \quad \nabla \times \vec{H}=\frac{\varepsilon}{c} \frac{\partial \vec{E}}{\partial t}+\frac{4 \pi \sigma}{c} \vec{E} . \tag{1.1.1}
\end{equation*}
$$

Here $\vec{E}$ is an electric vector, $\vec{H}$ is a magnetic vector, $\varepsilon$ is permittivity, $\sigma$ is electrical conductivity, $c$ is the velocity of light.

The $\vec{E}, \vec{H}$ field satisfies the equations (1.1.1) at the $\varepsilon$ and $\sigma$ continuity points. At the $\varepsilon$ and $\sigma$ discretion points, i.e., at the boundaries between the layers $\vec{E}$ and $\vec{H}$ tangential components ought to be continuous. A study of the field in a layered medium, in effect, comes down to the solution of equation (1.1.1) with specified boundary conditions when the field is excited by an incident plane wave.

Let us denote the space variable, perpendicular to the boundary of layers, as $z$ and direct the $\quad z$ axis outwardly, to where the incident wave (Fig. 1.1) comes from. Let the incident wave
be a plane monochromatic $\omega$-frequency wave. Let us assume a time dependence of the field be $\exp (i \omega t){ }^{1}$ and hold

$$
\vec{E}=\overrightarrow{\mathbf{E}} \exp (i \omega t), \quad \vec{H}=\overrightarrow{\mathbf{H}} \exp (i \omega t) .
$$

Here vectors $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$ depend only on the spatial variables.
Substituting these expressions into (1.1.1) and further reducing by $\exp (i \omega t)$ exponential factor, we obtain

$$
\begin{equation*}
\nabla \times \overrightarrow{\mathbf{E}}=-i \frac{\omega}{c} \overrightarrow{\mathbf{H}}, \quad \nabla \times \overrightarrow{\mathbf{H}}=i \frac{\omega}{c} \varepsilon \overrightarrow{\mathbf{E}}+\frac{4 \pi \sigma}{c} \overrightarrow{\mathbf{E}} . \tag{1.1.2}
\end{equation*}
$$

Typically the wavelength of incident wave in vacuum is used as a spectral parameter characterizing a monochromatic wave:

$$
\lambda=\frac{2 \pi c}{\omega} .
$$

Another convenient spectral parameter is the wavenumber of the incident wave in vacuum $k=\omega / c$, related to the wavelength by the equation:

$$
k=\frac{2 \pi}{\lambda} .
$$

It is also convenient for further use to introduce a complex permittivity of the medium defined as

$$
\tilde{\varepsilon}=\varepsilon-i \frac{4 \pi \sigma}{c}
$$

In conformity with the previously arranged direction of the $z$-axis, the complex permittivity turns to be a function of a variable $z$. Taking into account the above assignations, equations (1.1.2) will take the following form:

$$
\begin{equation*}
\nabla \times \overrightarrow{\mathbf{E}}=-i k \overrightarrow{\mathbf{H}}, \quad \nabla \times \overrightarrow{\mathbf{H}}=i k \tilde{\varepsilon}(z) \overrightarrow{\mathbf{E}} . \tag{1.1.3}
\end{equation*}
$$

Let us decompose the electromagnetic field of the wave into two components, the first being the $S$-component, with the electric vector perpendicular to the incidence plane, the other one, the $P$-component, featuring the plane-parallel electric vector. Let us select a coordinate system so that the $x$-axis be perpendicular to the plane of incidence and the $y$-axis be in this plane.

[^0]Let us denote the angle of incidence as $\gamma_{a}$, the permittivity of the outer space where the incident wave comes from as $\varepsilon_{a}$; the permittivity of the second outer space, traditionally referred to as substrate, as $\varepsilon_{S}$. Let us then consider equations for the $S$-component and the $P$-component of the field separately.

## The S-polarization case.

In this case, the electric vector $\overrightarrow{\mathbf{E}}$ has only one component $E_{x}$ different from 0.
Fig. 1.2 shows the $y z$-cross-section of the layered structure in question. The $x$-axis is directed backwards, so the $\overrightarrow{\mathbf{E}}$ vector is perpendicular to the $y z$-plane, away from the observer. The scalar form of the vector equations (1.1.3) gives:

$$
\begin{gather*}
H_{x}=0, \quad \frac{\partial E_{x}}{\partial z}=-i k H_{y}, \quad \frac{\partial E_{x}}{\partial y}=i k H_{z} \\
\frac{\partial H_{z}}{\partial y}-\frac{\partial H_{y}}{\partial z}=i k \tilde{\varepsilon}(z) E_{x}, \quad \frac{\partial H_{z}}{\partial x}=0, \quad \frac{\partial H_{y}}{\partial x}=0 \tag{1.1.4}
\end{gather*}
$$



Fig. 1.2. Electric and magnetic vectors orientation in the $S$ - and $P$-polarization cases.

The first equation shows that the magnetic vector has only $H_{y}$ and $H_{z}$ components different from 0 . Thus, the magnetic vector lies in the $y z$-plane, i.e., in the incidence plane. It is an evidence of the crosswise field of the plane electromagnetic wave, with the electric, the magnetic and the wave vectors forming the right-hand triple. It follows from the two other
equations that the magnetic vector, hence, the electric vector, as well, depend only on $y$ and $z$. Substituting $H_{y}$ and $H_{z}$ from the second and third equations into the fourth one, we obtain:

$$
\begin{equation*}
\frac{\partial^{2} E_{x}}{\partial y^{2}}+\frac{\partial^{2} E_{x}}{\partial z^{2}}+k^{2} \tilde{\varepsilon}(z) E_{x}=0 \tag{1.1.5}
\end{equation*}
$$

Let us apply the variable separation method to the equation, in other words let us seek its solution in the form:

$$
E_{x}(y, z)=u(z) g(y) .
$$

By substituting this expression into (1.1.5), dividing it by $u(z) g(y)$ and separating the variables, we obtain

$$
\begin{equation*}
-\frac{1}{g(y)} \frac{d^{2} g}{d y^{2}}=\frac{1}{u(z)}\left[\frac{d^{2} u}{d z^{2}}+k^{2} \tilde{\varepsilon}(z) u(z)\right] \tag{1.1.6}
\end{equation*}
$$

It follows from this equation that both its parts can depend neither on $y$ or on $z$ and, by this very fact, are equal to a certain constant value. This constant will be further determined from the condition of the field excitation by an incident plane wave. For the time being, it is convenient to assume it equal to $k^{2} \alpha^{2}$, where $k$ is the wavenumber of the incident wave, and $\alpha$ value is undetermined so far. Then we obtain the following equation for the $g(y)$ function from (1.1.6):

$$
\begin{equation*}
\frac{d^{2} g}{d y^{2}}+k^{2} \alpha^{2} g(y)=0 \tag{1.1.7}
\end{equation*}
$$

Let us select its solution in the following form:

$$
g(y)=\exp (i k \alpha y)
$$

This presentation does in no way affect its common character, as the numerical constant placed in a general case before the exponential, can be included into the $u(z)$ function and the sign of $\alpha$ has not yet been determined. So $E_{x}$ is written down in the following form

$$
E_{x}(y, z)=u(z) \exp (i k \alpha y)
$$

It follows from the second and third equations of the system (1.1.4) that $H_{y}$ and $H_{z}$ have a similar type of dependence on $y$. Then, we obtain from the third equation the following:

$$
H_{z}(y, z)=\alpha u(z) \exp (i k \alpha y) .
$$

Assume

$$
H_{y}(y, z)=-v(z) \exp (i k \alpha y) .
$$

Thus, the vector functions $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$ are written down as

$$
\begin{align*}
& \overrightarrow{\mathbf{E}}=\{u(z), 0,0\} \exp (i k \alpha y),  \tag{1.1.8}\\
& \overrightarrow{\mathbf{H}}=\{0,-v(z), \alpha u(z)\} \exp (i k \alpha y) .
\end{align*}
$$

Here the first, the third, the fifth and the sixth equations of the system (1.1.4) are satisfied. The remaining two equations provide a system for determining the $u(z)$ and $v(z)$ functions. By substituting the above expressions for the field components into the equations and reducing them by $\exp (i k \alpha y)$, we obtain

$$
\begin{equation*}
\frac{d u}{d z}=i k v, \quad \frac{d v}{d z}=i k\left[\tilde{\varepsilon}(z)-\alpha^{2}\right] u . \tag{1.1.9}
\end{equation*}
$$

The equations (1.1.9) are valid for the field at every continuity points of complex permittivity $\tilde{\varepsilon}(z)$, i.e., everywhere at the $\varepsilon$ and $\sigma$ continuity points. At the $\varepsilon$ and $\sigma$ discretion points, i.e., at the boundaries of the layers tangential components of the $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$ ought to be continuous. The $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$ tangential components are equal to $u(z) \exp (i k \alpha y)$ and $-v(z) \exp (i k \alpha y)$. Since the continuity conditions ought to be satisfied at the layer boundary at any $y$ value, it follows, then, that the $y$-dependence in any layer, in the outer space, and in the substrate is the same, while the $u(z)$ and $v(z)$ functions are continuous at the boundaries of layers.

Let us now determine the constant $\alpha$. The field of the incident plane wave can be presented as ${ }^{2}$

$$
\vec{E}=\overrightarrow{\mathbf{E}}_{A} \exp (-i \vec{k} \vec{r}+i \omega t) .
$$

Here $\overrightarrow{\mathbf{E}}_{A}$ is the field amplitude, $\vec{k}$ is the wave vector of the incident wave, $\vec{r}$ is the coordinate vector of the observation point.

In a homogeneous isotropic medium featuring $\varepsilon_{a}$ permittivity, the wave vector is equal to $\vec{k}=k \sqrt{\varepsilon_{a}} \vec{l}$, where $\vec{l}$ is a unit directing vector. Let the incident wave spread in the direction shown in Fig. 1.2. Then the directing vector is

$$
\vec{l}=\left\{0,-\sin \gamma_{a},-\cos \gamma_{a}\right\}
$$

[^1]and the incident wave field depends on the coordinates in the following way:
$$
\vec{E}=\overrightarrow{\mathbf{E}}_{A} \exp \left[i k \sqrt{\varepsilon_{a}}\left(z \cos \gamma_{a}+y \sin \gamma_{a}\right)+i \omega t\right] .
$$

Hence, we obtain

$$
\begin{equation*}
\alpha=\sqrt{\varepsilon_{a}} \sin \gamma_{a} \tag{1.1.10}
\end{equation*}
$$

This equation represents the Snell law. Actually, $\alpha$ is the same in all layers, the outer space, and the substrate. But in the latter $\alpha=\sqrt{\varepsilon_{s}} \sin \gamma_{s}$ and so

$$
\begin{equation*}
\sqrt{\varepsilon_{a}} \sin \gamma_{a}=\sqrt{\varepsilon_{s}} \sin \gamma_{s} \tag{1.1.11}
\end{equation*}
$$

The Snell law (1.1.11) is valid also in case of absorbing substrate. In this case complex permittivity $\tilde{\varepsilon}_{s}$, should be substituted into (1.1.11) instead of $\varepsilon_{s}$. The $\gamma_{s}$ angle in this case will also be complex, which, in the physical sense implies a non-coincidence of the plane of the constant wave phase with that of the constant wave amplitude.

## The P-polarization case.

Now the $\overrightarrow{\mathbf{H}}$ vector has only $H_{x}$ component different from zero, and the $\overrightarrow{\mathbf{E}}$ vector has two $E_{y}$ and $E_{z}$ components. The vector equations (1.1.3) written in the scalar form give the following system of equations:

$$
\begin{align*}
& \frac{\partial E_{z}}{\partial y}-\frac{\partial E_{y}}{\partial z}=-i k H_{x}, \quad \frac{\partial E_{z}}{\partial x}=0, \quad \frac{\partial E_{y}}{\partial x}=0  \tag{1.1.12}\\
& E_{x}=0, \quad \frac{\partial H_{x}}{\partial z}=i k \tilde{\varepsilon}(z) E_{y}, \quad \frac{\partial H_{x}}{\partial y}=-i k \tilde{\varepsilon}(z) E_{z} .
\end{align*}
$$

Similar to the $S$-case, we establish that the field components depend only on y and $z$ and the $y$-dependence takes the form of $\exp (i k \alpha y)$. Let us represent $H_{x}$ as

$$
H_{x}(y, z)=v(z) \exp (i k \alpha y) .
$$

From equations (1.1.12) it follows that

$$
E_{z}(y, z)=-\frac{\alpha}{\tilde{\varepsilon}(z)} v(z) \exp (i k \alpha y) .
$$

It is convenient to represent $E_{y}$ as

$$
E_{y}(y, z)=u(z) \exp (i k \alpha y)
$$

Thus, the $\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{H}}$ electromagnetic field in the $P$-case is written down as

$$
\begin{align*}
& \overrightarrow{\mathbf{E}}=\left\{0, u(z),-\frac{\alpha}{\tilde{\varepsilon}(z)} v(z)\right\} \exp (i k \alpha y),  \tag{1.1.13}\\
& \overrightarrow{\mathbf{H}}=\{v(z), 0,0\} \exp (i k \alpha y)
\end{align*}
$$

Substituting the field components into the first and the fifth equations of the system (1.1.12) and reducing it by $\exp (i k \alpha y)$, we obtain the equations for the unknown functions $u(z)$ and $v(z)$

$$
\begin{align*}
& \frac{d u}{d z}=i k\left[1-\frac{\alpha^{2}}{\tilde{\varepsilon}(z)}\right] v  \tag{1.1.14}\\
& \frac{d v}{d z}=i k \tilde{\varepsilon}(z) u
\end{align*}
$$

Like in the previous case, the equations are valid at $\varepsilon$ and $\sigma$ continuity points. At the boundaries of layers, the $u(z)$ and $v(z)$ functions continuity as well as the $\alpha$ constancy follow from the continuity of the field tangential components. Constant $\alpha$, as is easily seen, is still determined from the equation (1.1.10).

We will regard equations (1.1.9)-(1.1.14) as principal relations describing the field in the layered medium. Field $\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{H}}$ is determined from the solutions of these equations according to the formulas (1.1.8), (1.1.13).

Note, that in case of normal incidence, i.e., when $\alpha=\sqrt{\varepsilon_{a}} \sin \gamma_{a}=0$, equations (1.1.9) and (1.1.14) for $S$ - and $P$-components of the field coincide.

### 1.1.2 Amplitude transmittance and reflectance of the layered medium

Now let us try to obtain general expressions for the transmittance and reflectance of a layered medium. Here, we will again consider $S$ - and $P$ - polarization cases.

The $S$-polarization case.
Equations (1.1.9) are also valid for the outer space and the substrate. In the outer space $\varepsilon(z) \equiv \varepsilon_{a}$ and from (1.1.9) we obtain a simplest second order equation for $u(z)$ :

$$
\begin{equation*}
\frac{d^{2} u}{d z^{2}}+k^{2}\left(\varepsilon_{a}-\alpha^{2}\right) u=0 \tag{1.1.15}
\end{equation*}
$$

Assume

$$
q_{a}=\sqrt{\varepsilon_{a}-\alpha^{2}}=\sqrt{\varepsilon_{a}} \cos \gamma_{a} .
$$

Equation (1.1.15) has two linearly independent solutions: $\exp \left(-i k q_{a} z\right)$ and $\exp \left(i k q_{a} z\right)$. Selecting $\exp (i \omega t)$ as the time exponential factor, the first one corresponds to the wave spreading in the positive direction of the $z$-axis while the other one represents the negative direction of the wave propagation along the $z$-axis. So, for the selected coordinate system, dependence of the incident wave field on $z$ takes the form $\operatorname{of} \exp \left(i k q_{a} z\right)$, while the reflected wave field dependence looks as $\exp \left(-i k q_{a} z\right)$. For the incident wave it follows from the first equation in (1.1.9) that $v=q_{a} u$, while for the reflected wave we obtain $v=-q_{a} u$.

Let us designate the coordinate of the outer space boundary as $z_{a}$ and denote amplitudes of tangential components of the electric vector of the incident and reflected waves at this boundary as $E_{A}$ and $E_{R}$. Thus, for the incident wave

$$
u\left(z_{a}\right)=E_{A}, \quad v\left(z_{a}\right)=q_{a} E_{A} .
$$

and for the reflected wave

$$
u\left(z_{a}\right)=E_{R}, \quad v\left(z_{a}\right)=-q_{a} E_{R} .
$$

Similarly, designating $E_{T}$ the tangential component of the electric vector of the transmitted wave at the boundary with substrate we have

$$
u(0)=E_{T}, \quad v(0)=q_{s} E_{T},
$$

where

$$
q_{s}=\sqrt{\varepsilon_{s}-\alpha^{2}}=\sqrt{\varepsilon_{s}} \cos \gamma_{s} .
$$

Amplitude transmittance and reflectance are introduced for the $S$-component of the field as follows

$$
t=\frac{E_{T}}{E_{A}}, \quad r=\frac{E_{R}}{E_{A}} .
$$

The amplitude coefficients do not change with proportional changes of the amplitudes of the incident, reflected and transmitted waves. So, the fields can be normalized with respect to the amplitude of the transmitted field. Then $E_{T}$ is equal to 1.

The equations (1.1.9) solutions do not only depend on $z$, but also on the wavenumber $k$. So, further we will take them down as $u(z, k), v(z, k)$ pointing out their dependence on the wavenumber $k$ as well as on the coordinate $z$.

The continuity of the $u$ and $v$ functions at the boundary with the substrate leads to the followings initial conditions for equations

$$
\begin{equation*}
u(0, k)=1, \quad v(0, k)=q_{s} . \tag{1.1.16}
\end{equation*}
$$

The field at the outer boundary of the layered media $z_{a}$ is the sum of the incident wave and the reflected wave fields. So, the continuity conditions for $u$ and $v$ at the outer space boundary and the above relations for the incident and the reflected fields amplitudes lead to:

$$
u\left(z_{a}, k\right)=E_{A}+E_{R}, \quad v\left(z_{a}, k\right)=q_{a}\left(E_{A}-E_{R}\right) .
$$

These relations allow to express amplitudes of the incident and reflected fields through the solution of system (1.1.9) with the initial conditions (1.1.16):

$$
E_{R}=\frac{q_{a} u\left(z_{a}, k\right)-v\left(z_{a}, k\right)}{2 q_{a}}, \quad E_{A}=\frac{q_{a} u\left(z_{a}, k\right)+v\left(z_{a}, k\right)}{2 q_{a}} .
$$

Note, these will be $E_{R}$ and $E_{A}$ values for $E_{T}=1$. This immediately leads to expressions for the amplitude transmittance and reflectance. They depend on the wavenumber $k$ and are determined from

$$
\begin{equation*}
t(k)=\frac{2 q_{a}}{q_{a} u\left(z_{a}, k\right)+v\left(z_{a}, k\right)}, \quad r(k)=\frac{q_{a} u\left(z_{a}, k\right)-v\left(z_{a}, k\right)}{q_{a} u\left(z_{a}, k\right)+v\left(z_{a}, k\right)} . \tag{1.1.17}
\end{equation*}
$$

Thus, normalizing the field with respect to the amplitude of transmitted wave allows to bring amplitude reflectance and transmittance calculations to finding the solution of the differential equations (1.1.9) with the initial conditions (1.1.16). We will discuss in detail the calculating aspects of the technique a little later. Meanwhile, we will consider the $P$-component of the field.

## The P-polarization case.

From equations (1.1.14) for $u(z)$ in the outer space we again obtain equation (1.1.15). So, in the $P$-case, similar considerations on dependence of the incident and the reflected waves on $z$ as in $S$-case are valid.

So, for the incident wave

$$
u(z) \sim \exp \left(i k \sqrt{\varepsilon_{a}-\alpha^{2}} z\right)
$$

while for the reflected wave

$$
u(z) \sim \exp \left(-i k \sqrt{\varepsilon_{a}-\alpha^{2}} z\right)
$$

Taking this into account, we obtain from the second equation of the system (1.1.14) that for the incident wave $v=q_{a} u$, while for the reflected one $v=-q_{a} u$, where the value $q_{a}$ is now determined as

$$
q_{a}=\frac{\varepsilon_{a}}{\sqrt{\varepsilon_{a}-\alpha^{2}}}=\frac{\sqrt{\varepsilon_{a}}}{\cos \gamma_{a}} .
$$

Similarly, for the transmitted wave $v=q_{s} u$, where

$$
q_{s}=\frac{\varepsilon_{s}}{\sqrt{\varepsilon_{s}-\alpha^{2}}}=\frac{\sqrt{\varepsilon_{s}}}{\cos \gamma_{s}}
$$

Like in the S-case, we will determine the amplitude transmittance and reflectance through the amplitudes of the tangential components of the electric vector of the incident, reflected and transmitted waves $E_{A}, E_{R}$ and $E_{T}$, respectively. In accordance with (1.1.13) and the equations connecting the $u$ and $v$ functions in the outer space and the substrate, we have for the incident wave

$$
u\left(z_{a}\right)=E_{A}, \quad v\left(z_{a}\right)=q_{a} E_{A}
$$

for the reflected wave

$$
u\left(z_{a}\right)=E_{R}, \quad v\left(z_{a}\right)=-q_{a} E_{R}
$$

and for the transmitted wave

$$
u(0)=E_{T}, \quad v(0)=q_{s} E_{T} .
$$

Let us again normalize the fields with respect to the transmitted wave amplitude, assuming $E_{T}=1$. Let $u(z, k), v(z, k)$ be solutions for system (1.1.14) in the layered medium. The $u$ and $v$ functions continuity at the boundary with substrate leads us to the initial conditions for system (1.1.14) in the form:

$$
u(0, k)=1, \quad v(0, k)=q_{s},
$$

which looks like initial conditions (1.1.16). However, the value $q_{s}$, is calculated from another formula for the $P$-case (see above). The continuity condition at the boundary with the outer space implies that

$$
u\left(z_{a}, k\right)=E_{A}+E_{R}, \quad v\left(z_{a}, k\right)=q_{a}\left(E_{A}-E_{R}\right)
$$

Expressing $E_{A}$ and $E_{R}$ with the help of $u\left(z_{a}, k\right)$ and $v\left(z_{a}, k\right)$, and taking account of $E_{T}=1$, we again obtain (1.1.17) for the amplitude transmittance and reflectance.

So, the calculation of amplitude coefficients in the $P$-case also comes down to the solution of differential equations (in this instance, it is the system (1.1.14)).

Let us consider formulas (1.1.17) in greater detail. Note, that they solve the problem of calculating amplitude transmittance and reflectance for the layered medium of arbitrary type, either consisting of homogeneous layers or the one consisting of inhomogeneous layers with continuous or discretely continuous distribution of parameters. The layered medium can be absorbing. The substrate can also feature absorbing properties, in which case the $q_{s}$ value in the initial conditions will be complex.

Let us also note, that so far, most works employ the following technique for calculating the amplitude transmittance and reflectance for the optical coating with inhomogeneous layers. The inhomogeneous layers are broken into a great number (sometimes as many as thousands) of homogeneous layers, approximating a smooth curve of the parameter distribution. Then the amplitude coefficients are calculated by way of recurrent formulas (see section 1.2). This technique is in no way obligatory since any modern computer has standard software for differential equations systems solutions. It is erroneous to believe that employing recurrent formulas here prove to be more time-efficient. In fact, the situation is reverse. Any recurrent formulas require numerous calculations of trigonometric functions or of the complex argument exponentials, which, in their turn, result in a significant number of arithmetic operations. At the same time, standard software for differential equations systems employ but simplest arithmetic operations. They are all optimized in time-consumption. Employing differential equations (1.1.9), (1.1.14) for calculating the amplitude transmittance and reflectance of the optical coating with continuously changing parameters is also preferable from the point of view of the calculation precision. The step-like approximation of the parameters allow only the first order precision in calculations. At the same time standard software for differential equations systems employ methods of the fourth and higher order of precision.

Systems of equations (1.1.9), (1.1.14) also allow to calculate the field distribution in the layered medium with an arbitrary dependence $\tilde{\varepsilon}(z)$.

In order to calculate the amplitude coefficients of the medium consisting of homogeneous layers, various recurrent calculating techniques are preferable. They will be considered in the next section of the chapter.

For the purpose of convenience, let us congregate all the results obtained above. The amplitude transmittance and reflectance in the $S$ - and $P$-cases are calculated as

$$
t(k)=\frac{2 q_{a}}{q_{a} u\left(z_{a}, k\right)+v\left(z_{a}, k\right)}, \quad r(k)=\frac{q_{a} u\left(z_{a}, k\right)-v\left(z_{a}, k\right)}{q_{a} u\left(z_{a}, k\right)+v\left(z_{a}, k\right)} .
$$

where $u(z, k), v(z, k)$ functions are solutions of the differential equations system (1.1.9) in the $S$-case

$$
\frac{d u}{d z}=i k v, \quad \frac{d v}{d z}=i k\left[\tilde{\varepsilon}(z)-\alpha^{2}\right] u
$$

and the differential equations system (1.1.14):
in the P-case

$$
\frac{d u}{d z}=i k\left[1-\frac{\alpha^{2}}{\tilde{\varepsilon}(z)}\right] v, \quad \frac{d v}{d z}=i k \tilde{\varepsilon}(z) u .
$$

The initial conditions for the differential equations are set at the boundary with the substrate as

$$
u(0, k)=1, \quad v(0, k)=q_{s} .
$$

The $\alpha, q_{a}, q_{s}$ parameters are calculated from:

$$
\alpha=\sqrt{\varepsilon_{a}} \sin \gamma_{a}
$$

in the S-case:

$$
q_{a}=\sqrt{\varepsilon_{a}-\alpha^{2}}, \quad q_{s}=\sqrt{\varepsilon_{s}-\alpha^{2}}
$$

in the P-case:

$$
q_{a}=\frac{\varepsilon_{a}}{\sqrt{\varepsilon_{a}-\alpha^{2}}}, \quad q_{s}=\frac{\varepsilon_{s}}{\sqrt{\varepsilon_{s}-\alpha^{2}}} .
$$

In many cases it is more convenient to employ the concept of the refractive index than that of the permittivity of the optical medium. The permittivity $\varepsilon$ is connected with the refractive index $n$ through the equation

$$
\varepsilon=n^{2} .
$$

If it is an absorbing medium, its refractive index, like its permittivity, is complex. We will also denote it with a wave line over the symbol.

Thus, for the general case

$$
\tilde{\varepsilon}(z)=\tilde{n}^{2}(z)
$$

Let us introduce the refractive indices of the outer space and the substrate as $n_{a}=\sqrt{\varepsilon_{a}}$, and $n_{s}=\sqrt{\varepsilon_{s}}$ respectively. Value of $\alpha$ can now be expressed as

$$
\alpha=n_{a} \sin \gamma_{a}
$$

and the Snell law can be put down as

$$
n_{a} \sin \gamma_{a}=n_{s} \sin \gamma_{s} .
$$

The $q_{a}$ and $q_{s}$ parameters are also conveniently expressed through corresponding refractive indices and wave propagation angles $\gamma_{a}$ and $\gamma_{s}$. Simplest arithmetic operations and the Snell law application permit to obtain in the $S$-case:

$$
q_{a}=n_{a} \cos \gamma_{a}, \quad q_{s}=n_{s} \cos \gamma_{s}
$$

in the P-case:

$$
q_{a}=\frac{n_{a}}{\cos \gamma_{a}}, \quad q_{s}=\frac{n_{s}}{\cos \gamma_{s}}
$$

### 1.1.3 The Fresnel formulas

The results of the above considerations allow to easily obtain the Fresnel formulas for the amplitude transmittance and reflectance for the boundary between two media. Assume $z_{a}=0$, then the layered medium under consideration will be reduced to the boundary between the outer space and the substrate. Here $u\left(z_{a}, k\right)=u(0, k)=1, v\left(z_{a}, k\right)=v(0, k)=q_{s}$ and the following expressions are obtained for the amplitude coefficients:

$$
t=\frac{2 q_{a}}{q_{a}+q_{s}}, \quad r=\frac{q_{a}-q_{s}}{q_{a}+q_{s}} .
$$

Substituting $q_{a}, q_{s}$, expressed through refractive indices of the media and the wave propagation angles, we obtain
in the $S$-case:

$$
t=\frac{2 n_{a} \cos \gamma_{a}}{n_{a} \cos \gamma_{a}+n_{s} \cos \gamma_{s}}, \quad r=\frac{n_{a} \cos \gamma_{a}-n_{s} \cos \gamma_{s}}{n_{a} \cos \gamma_{a}+n_{s} \cos \gamma_{s}},
$$

in the P-case:

$$
t=\frac{2 n_{a} \cos \gamma_{s}}{n_{a} \cos \gamma_{s}+n_{s} \cos \gamma_{a}}, \quad r=\frac{n_{a} \cos \gamma_{s}-n_{s} \cos \gamma_{a}}{n_{a} \cos \gamma_{s}+n_{s} \cos \gamma_{a}} .
$$

Those are, in fact, the Fresnel formulas for the amplitude transmittance and reflectance for the boundary between the media featuring the $n_{a}$ and $n_{s}$ refractive indices when the incident wave comes from the medium with refractive index $n_{a}$ at the incidence angle $\gamma_{a}$.

Note, that very often, the formula for the amplitude reflectance in the $P$-case is written down with the opposite sign. This result can be obtained if in both cases the amplitude reflectance is determined as the ratio of the field components perpendicular to the plane of incidence (being the electric vector components in the $S$-case, and the magnetic vector components in the $P$-case). We, however, determined the amplitude reflectance as the ratio of the tangential components of the electric vector for both cases. It seems a more proper way since it will eventually lead to coincidence of both amplitude reflectance expressions at the normal incidence of light when the difference between the $S$ - and the $P$-components disappears.

### 1.1.4 Other ways of determining amplitude transmittance and reflectance of the layered medium

In a number of cases, other ways of determining amplitude transmittance and reflectance are preferable for the purpose of calculation and analytical investigation of their properties.

In case of a non-absorbing medium or when the amplitude reflectance alone is sufficient for further use, it is convenient to employ a differential equation for a direct determination of $r(k)$. Let us introduce the so-called local reflection function

$$
\begin{equation*}
r(z, k)=\frac{q_{a} u(z, k)-v(z, k)}{q_{a} u(z, k)+v(z, k)} . \tag{1.1.18}
\end{equation*}
$$

Here $u(z, k)$ and $v(z, k)$ are solutions of equations (1.1.9) or (1.1.14) for the $S$ - and $P$-cases, respectively. In the $r(z, k)$ function arguments we, like in the cases above, reveal its dependence on the wavenumber $k$. As is seen from the definition and formulas (1.1.17), the physical meaning of the local reflection function is revealed as follows. It gives the amplitude reflectance of the part of the layered medium confined within 0 and $z$ coordinates on condition that a homogeneous outer space featuring $\varepsilon_{a}$ permittivity is found in the area to the right of $z$. It is evident that with $z=z_{a}$ the local reflection function coincides with the amplitude reflectance for the whole of the layered medium:

$$
r(k)=r\left(z_{a}, k\right)
$$

By differentiating expression (1.1.18) and employing equations (1.1.9) and (1.1.14), we obtain after a few non-complicated operations
for the S-case

$$
\begin{equation*}
\frac{d r}{d z}=\frac{i k}{2}\left\{q_{a}(1-r)^{2}-\frac{1}{q_{a}}\left[\tilde{\varepsilon}(z)-\alpha^{2}\right](1+r)^{2}\right\}, \tag{1.1.19}
\end{equation*}
$$

for the S-case

$$
\begin{equation*}
\frac{d r}{d z}=\frac{i k}{2}\left\{q_{a}\left[1-\frac{\alpha^{2}}{\tilde{\varepsilon}(z)}\right](1-r)^{2}-\frac{\tilde{\varepsilon}(z)}{q_{a}}(1+r)^{2}\right\} . \tag{1.1.20}
\end{equation*}
$$

We obtain the initial conditions for these equations assuming $z=0$ in (1.1.18) and employing the initial conditions (1.1.16) for $u$ and $v$. Here, for both polarizations

$$
r(0, k)=\frac{q_{a}-q_{s}}{q_{a}+q_{s}}
$$

but, like before, $q_{a}$ and $q_{s}$ differ for the $S$ - and $P$-cases.
Equations (1.1.19), (1.1.20) are Riccati-type nonlinear differential equations. However, from the point of view of their numerical solution it involves no further difficulties compared to the linear equations (1.1.9), (1.1.14) solutions. On the contrary, if only the amplitude reflectance is sufficient, it is preferable to use equations (1.1.19), (1.1.20) as it involves only one complex equation for solution (or a system of two real) rather than a system of two complex equations (or four real ones), like in the case above. As to the rest, the general remarks we made at the end of 1.1.2 subsection, remain valid here.

In this way we can try to obtain the "local amplitude transmittance" equation. However, it is not worthwhile as no independent equation can be obtained for $t$ : besides $t$ it will also include the $r$-function of local reflection. So, if it becomes necessary to determine the amplitude transmittance, it is reasonable to use systems (1.1.9), (1.1.14) directly.

### 1.1.5 Transmittance, reflectance and absorptance

The magnitude and the direction of energy flow for the harmonic wave is determined by the time-averaged Poynting vector. It is found from the formula (see Born and Wolf, 1959):

$$
\vec{S}=\frac{c}{8 \pi} \operatorname{Re}\left[\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{H}}^{*}\right]
$$

The asterisk here is a symbol of complex conjugation, the square brackets denote the vector product. The energy flow along the normal toward the layered medium is determined by
the $z$-component of the Poynting vector. We obtain from (1.1.8), (1.1.13) for the $S$ - and $P$-cases, that

$$
\begin{equation*}
\vec{S}=\frac{c}{8 \pi} \operatorname{Re}\left(-u v^{*}\right) \tag{1.1.21}
\end{equation*}
$$

Assume again, that $E_{A}, E_{R}$ and $E_{T}$ are amplitudes of the tangential components of the electric vector of the incident, reflected and transmitted waves. For the incident wave $u\left(z_{a}\right)=E_{A}, v\left(z_{a}\right)=q_{a} E_{A}$ at the outer boundary of the layered medium. So, in compliance with formula (1.1.21) for the incident wave, the energy flow along the normal toward the layered medium at the outer boundary is proportional to

$$
-\operatorname{Re}\left\{q_{a}\right\}\left|E_{A}\right|^{2}
$$

The negative sign here indicates that the energy propagates in the negative direction of the $z$-axis. Similarly, taking into account that for the reflected wave $u\left(z_{a}\right)=E_{R}, v\left(z_{a}\right)=-q_{a} E_{R}$, we obtain the energy flow for the reflected wave along the $z$-axis which is proportional to

$$
\operatorname{Re}\left\{q_{a}\right\}\left|E_{R}\right|^{2} .
$$

The reflectance of the layered medium is determined as the ratio of absolute values of these flows and is, evidently, equal to

$$
\begin{equation*}
R=\left|\frac{E_{R}}{E_{A}}\right|^{2}=|r|^{2}, \tag{1.1.22}
\end{equation*}
$$

where $r$ is the amplitude reflectance.
Similarly, we find that the transmitted wave energy flow along the $z$-axis is proportional to

$$
-\operatorname{Re}\left\{q_{s}\right\}\left|E_{T}\right|^{2} .
$$

The transmittance is introduced as a ratio of the absolute value of this flow to the absolute value of the energy flow of the incident wave along the $z$-axis, and thus is equal to

$$
\begin{equation*}
T=\frac{\operatorname{Re}\left\{q_{s}\right\}}{\operatorname{Re}\left\{q_{a}\right\}}\left|\frac{E_{T}}{E_{A}}\right|^{2}=\frac{\operatorname{Re}\left\{q_{s}\right\}}{\operatorname{Re}\left\{q_{a}\right\}}|t|^{2}, \tag{1.1.23}
\end{equation*}
$$

where $t$ is the amplitude transmittance. In the case of non-absorbing outer space and substrate $\operatorname{Re}\left\{q_{a}\right\}=q_{a}, \operatorname{Re}\left\{q_{s}\right\}=q_{s}$.

Alongside with the transmittance and the reflectance, absorptance $A$ is introduced. It is determined as a part of the incident wave energy absorbed in the layered medium. The energy conservation law results in the equation

$$
A=1-R-T .
$$

Let the incident light be linear polarized in the plane having an angle $\psi$ with the incident plane. In this case the reflectance and transmittance are determined from the formulas

$$
\begin{equation*}
R=R_{s} \sin ^{2} \psi+R_{p} \cos ^{2} \psi, \quad T=T_{s} \sin ^{2} \psi+T_{p} \cos ^{2} \psi \tag{1.1.24}
\end{equation*}
$$

where $R_{s}, R_{p}, T_{s}, T_{p}$ are reflectances and transmittances for the $S$ - and $P$-components of the field. In the case of non-polarized incident light, expression (1.1.24) is averaged with respect to the $\psi$ angle. As a result, we obtain

$$
\begin{equation*}
R=\frac{1}{2}\left(R_{s}+R_{p}\right), \quad T=\frac{1}{2}\left(T_{s}+T_{p}\right) . \tag{1.1.25}
\end{equation*}
$$

### 1.2 Principal recurrent methods for calculating layered media spectral coefficients

When solving various problems in multilayer optics field, we assume in most cases that a layered medium consists of a finite number of homogeneous and isotropic layers. In this case in order to determine the amplitude transmittance and reflectance, it is convenient and efficient mathematically to employ various recurrent calculating techniques. Depending on the specific problem to be solved, a specific technique is preferred. Mastering various methods expands one's concept of the properties of multilayer coatings. The application of these methods will further permit to obtain a number of significant conclusions for analysis and synthesis.

### 1.2.1 Matrix calculating method

The matrix method for calculating spectral coefficients of the layered media was first suggested by F. Abeles (1950) and has been widely employed ever since. The results of the previous section allow to provide here its very simple presentation.

Let us assume a multilayer coating consisting of a finite number of homogeneous and isotropic layers (see Fig. 1.3). The figure shows that the same orientation of the $z$-axis with respect to the outer space and substrate as in the previous section is maintained. Let the total
number of the layers be equal to $m$. We will number the layers in the direction from the substrate to the outer space and denote their right boundaries as $z_{1}, z_{2}, \ldots$. Thus, $z_{m}=z_{a}$, where $z_{a}$ is a coordinate of the outer boundary of the coating (at the same time $z_{a}$ is a total geometrical thickness of the multilayer coating). Let us denote the geometrical thicknesses of the layers as $d_{1}, d_{2}, \ldots$, and their complex refractive indices as $\tilde{n}_{1}, \tilde{n}_{2}, \ldots$. Let us also denote the refractive indices of the substrate and of the outer space as $n_{s}$, and $n_{a}$, respectively. The substrate refractive index may also be complex, thus the substrate can be absorbing. When evolving the formulas of the preceding section, it was convenient to use the complex permittivity of media. The complex refractive indices employed here are connected with $\tilde{\varepsilon}$ by the equation

$$
\tilde{n}^{2}=\tilde{\varepsilon}
$$

As was shown earlier, the amplitude transmittance and reflectance can be determined in general cases by formula (1.1.17) using the solutions of the system of differential equations (1.1.9) in the $S$-case and (1.1.14) in the $P$-case. We will keep in mind that within each homogeneous layer, the solutions for these equations are written out explicitly. Let us first consider the S-component of the field. Equations (1.1.9) in the $j$-th layer are written down as follows:

$$
\begin{equation*}
\frac{d u}{d z}=i k v, \quad \frac{d v}{d z}=i k\left[\tilde{n}_{j}^{2}-\alpha^{2}\right] u \tag{1.2.1}
\end{equation*}
$$



Fig. 1.2. Multilayer coating including $m$ homogeneous layers.

From (1.2.1) we obtain the following differential equation of the second order for $u$ :

$$
\begin{equation*}
\frac{d^{2} u}{d z^{2}}+k^{2}\left[\tilde{n}_{j}^{2}-\alpha^{2}\right] u=0 \tag{1.2.2}
\end{equation*}
$$

It is convenient for further calculations to introduce the wave propagation angle in the $j$-th layer and denote it as $\gamma_{j}$. It comes from the Snell law

$$
\begin{equation*}
\tilde{n}_{j} \sin \gamma_{j}=\alpha=n_{a} \sin \gamma_{a} . \tag{1.2.3}
\end{equation*}
$$

Note, that $\gamma_{j}$ is a complex value in the absorbing layer. Taking account of (1.2.3), equation (1.2.2) can also be written down as follows

$$
\begin{equation*}
\frac{d^{2} u}{d z^{2}}+\left(k \tilde{n}_{j} \cos \gamma_{j}\right)^{2} u=0 \tag{1.2.4}
\end{equation*}
$$

Equation (1.2.4) has two linear independent solutions.lt is convenient to put them down as

$$
u_{1}(z)=\cos \left[k \tilde{n}_{j} \cos \gamma_{j}\left(z-z_{j-1}\right)\right], \quad u_{2}(z)=\sin \left[k \tilde{n}_{j} \cos \gamma_{j}\left(z-z_{j-1}\right)\right]
$$

with $z_{j-1}$ being the left boundary of the $j$-th layer.
The general solution of equation (1.2.4) is their linear combination:

$$
u(z)=c_{1} \cos \left[k \tilde{n}_{j} \cos \gamma_{j}\left(z-z_{j-1}\right)\right]+c_{2} \sin \left[k \tilde{n}_{j} \cos \gamma_{j}\left(z-z_{j-1}\right)\right] .
$$

Assuming $z=z_{j-1}$, we find $c_{1}=u\left(z_{j-1}\right)$. Differentiating with respect to $z$ and assuming again $z=z_{j-1}$, we obtain $k \tilde{n}_{j} \cos \gamma_{j} c_{2}=u^{\prime}\left(z_{j-1}\right)$ or taking account of the first equation in (1.2.1), we obtain $c_{2}=i v\left(z_{j-1}\right) /\left(\tilde{n}_{j} \cos \gamma_{j}\right)$. Thus, the solution of equation (1.2.4) in the $j$-th layer takes the form

$$
\begin{equation*}
u(z)=u\left(z_{j-1}\right) \cos \left[k \tilde{n}_{j} \cos \gamma_{j}\left(z-z_{j-1}\right)\right]+i v\left(z_{j-1}\right) \sin \left[k \tilde{n}_{j} \cos \gamma_{j}\left(z-z_{j-1}\right)\right] /\left(\tilde{n}_{j} \cos \gamma_{j}\right) \tag{1.2.5}
\end{equation*}
$$

Differentiating $u(z)$ function and substituting its derivative into the first equation of (1.2 .1 ), we obtain

$$
\begin{equation*}
v(z)=i \tilde{n}_{j} \cos \gamma_{j} u\left(z_{j-1}\right) \sin \left[k \tilde{n}_{j} \cos \gamma_{j}\left(z-z_{j-1}\right)\right]+v\left(z_{j-1}\right) \cos \left[k \tilde{n}_{j} \cos \gamma_{j}\left(z-z_{j-1}\right)\right] . \tag{1.2.6}
\end{equation*}
$$

Assuming $z=z_{j}$ in (1.2.5) and (1.2.6), we obtain the equations

$$
\begin{equation*}
u\left(z_{j}\right)=u\left(z_{j-1}\right) \cos \varphi_{j}+\left(i / q_{j}\right) v\left(z_{j-1}\right) \sin \varphi_{j}, \quad v\left(z_{j}\right)=i q_{j} u\left(z_{j-1}\right) \sin \varphi_{j}+v\left(z_{j-1}\right) \cos \varphi_{j}, \tag{1.2.7}
\end{equation*}
$$

where the following designations are introduced:

$$
\varphi_{j}=k \tilde{n}_{j} \cos \gamma_{j} d_{j}, \quad d_{j}=z_{j}-z_{j-1}, \quad q_{j}=\tilde{n}_{j} \cos \gamma_{j}
$$

The $\varphi_{j}$ is known as an angular phase thickness of the $j$-th layer.
Equations (1.2.7) can also be written down in the matrix form

$$
\binom{u}{v}_{z=z_{j}}=\left(\begin{array}{ll}
\cos \varphi_{j} & \left(i / q_{j}\right) \sin \varphi_{j}  \tag{1.2.8}\\
i q_{j} \sin \varphi_{j} & \cos \varphi_{j}
\end{array}\right)\binom{u}{v}_{z=z_{j-1}}
$$

The matrix introduced in (1.2.8), i.e.,

$$
M_{j}=\left(\begin{array}{ll}
\cos \varphi_{j} & \left(i / q_{j}\right) \sin \varphi_{j}  \tag{1.2.9}\\
i q_{j} \sin \varphi_{j} & \cos \varphi_{j}
\end{array}\right) .
$$

is referred to as the characteristic matrix of the $j$-th layer. According to (1.2.8), it allows recalculating the field from one boundary of the $j$-th layer to the other one.

Since the $u, v$ functions are continuous at the boundaries between the layers, we can recalculate the field from its boundary with the substrate $z_{0}=0$ to the boundary with the outer space $z_{m}=z_{a}$ making use of the characteristic matrices and formulas (1.2.8). In fact, by doing so, we find a solution of the differential equations for the field in the layered medium. So, we obtain

$$
\begin{equation*}
\binom{u}{v}_{z=z_{a}}=M_{m} M_{m-1} \ldots M_{1}\binom{u}{v}_{z=0} . \tag{1.2.10}
\end{equation*}
$$

The product of the characteristic matrices of the layers

$$
\begin{equation*}
M=M_{m} M_{m-1} \ldots M_{1} \tag{1.2.11}
\end{equation*}
$$

is known as the characteristic matrix of the multilayer coating. Let us denote its elements $m_{i j}$ :

$$
M=\left(\begin{array}{ll}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{array}\right)
$$

Mind that the co-factors in (1.2.11) are taken in the direction from the outer space towards the substrate.

Formulas (1.1.17) for the amplitude transmittance and reflectance include special solutions of the differential equations system featuring initial conditions

$$
\begin{equation*}
u(0, k)=1, \quad v(0, k)=q_{s}, \tag{1.2.12}
\end{equation*}
$$

where $q_{s}=\sqrt{\varepsilon_{s}-\alpha^{2}}=n_{s} \cos \gamma_{s}$. The characteristic matrix of a multilayer coating allows to find these solutions. Substituting the initial conditions (1.2.12) into the right part of (1.2.10) we obtain

$$
\begin{aligned}
& u\left(z_{a}, k\right)=m_{11}+m_{12} q_{s}, \\
& v\left(z_{a}, k\right)=m_{21}+m_{22} q_{s} .
\end{aligned}
$$

Substituting these solutions then into (1.1.17), we find that

$$
\begin{align*}
& t(k)=\frac{2 q_{a}}{q_{a} m_{11}+q_{s} m_{22}+q_{a} q_{s} m_{12}+m_{21}},  \tag{1.2.13}\\
& r(k)=\frac{q_{a} m_{11}-q_{s} m_{22}+q_{a} q_{s} m_{12}-m_{21}}{q_{a} m_{11}+q_{s} m_{22}+q_{a} q_{s} m_{12}+m_{21}}, \tag{1.2.14}
\end{align*}
$$

where $q_{a}=\sqrt{\varepsilon_{a}-\alpha^{2}}=n_{a} \cos \gamma_{a}$.
These expressions give the final formulas of the Abeles matrix method. In order to use them for calculating the amplitude transmittance and reflectance it is necessary to calculate characteristic matrices of the layers and find their product (1.2.11) which is the characteristic matrix of the multilayer coating.

In the P-polarization case, similar manipulations result again in formulas (1.2.13), (1.2.14) for amplitude transmittance and reflectance. The only difference here is the fact that the $q_{j}$ values in the characteristic matrices of the layers (1.2.9) are determined in $P$-case by the equations $q_{j}=\tilde{n}_{j} / \cos \gamma_{j}$, while $q_{s}, q_{a}$ in (1.2.13), (1.2.14) are equal to $n_{s} / \cos \gamma_{s}$ and $n_{a} / \cos \gamma_{a}$, respectively.

The transmittance and reflectance are determined through amplitude coefficients in accordance with the formulas:

$$
T=\frac{q_{s}}{q_{a}}|t|^{2}, \quad R=|r|^{2}
$$

For the sake of the user's convenience, Table 1.1 presents parameters included into calculating formulas (1.2.9), (1.2.13) and (1.2.14). The angles presented in Table 1.1 are determined from the Snell law:

$$
n_{s} \sin \gamma_{s}=\tilde{n}_{j} \sin \gamma_{j}=n_{a} \sin \gamma_{a},
$$

where $\gamma_{a}$ is the angle of incidence (a given value).

|  | $q_{s}$ | $q_{a}$ | $q_{j}$ | $\varphi_{j}$ |
| :---: | :---: | :---: | :---: | :---: |
| S-case | $n_{s} \cos \gamma_{s}$ | $n_{a} \cos \gamma_{a}$ | $\tilde{n}_{j} \cos \gamma_{j}$ | $\varphi_{j}=k \tilde{n} \cos \gamma_{j} d_{j}$ |
| P-case | $n_{s} / \cos \gamma_{s}$ | $n_{a} / \cos \gamma_{a}$ | $\tilde{n}_{j} / \cos \gamma_{j}$ |  |

Table 1.1: The calculation formulas parameters
With absorption absent, the refractive indices, $\gamma_{j}$-angles, and all the parameters presented in Table 1.1 are real. So the elements on the principal diagonals of the characteristic matrices of the layers will be real, while those on the subsidiary diagonals will be imaginary. By multiplying the matrices it is easy to check that the characteristic matrices of the coating will feature the same property. Thus, the first two items in the denominator of the expression for $t$ are real, while the two others are imaginary. It means that the transmittance for a non-absorbing coating may be put down as:

$$
T=\frac{4 q_{a} q_{s}}{\left(q_{a} m_{11}+q_{s} m_{22}\right)^{2}-\left(m_{21}+q_{a} q_{s} m_{12}\right)^{2}} .
$$

The negative sign here compensates the square of the imaginary unit included into the second brackets in the denominator. There is no need for a separate reflectance formula, as $R=1-T$.

The matrix method permits an easy calculation of the electromagnetic field distribution in the layered medium. Formulas (1.2.5) and (1.2.6) determine the tangential components of the electric and magnetic vectors of the $S$-component of the field at some arbitrary point of $j$-th layer. Similar formulas exist for the $P$-component of the field. Together they can be presented in the common form as:

$$
\begin{aligned}
& u(z)=u\left(z_{j-1}\right) \cos \left[k \tilde{n}_{j} \cos \gamma_{j}\left(z-z_{j-1}\right)\right]+\left(i / q_{j}\right) v\left(z_{j-1}\right) \sin \left[k \tilde{n}_{j} \cos \gamma_{j}\left(z-z_{j-1}\right)\right], \\
& v(z)=i q_{j} u\left(z_{j-1}\right) \sin \left[k \tilde{n}_{j} \cos \gamma_{j}\left(z-z_{j-1}\right)\right]+v\left(z_{j-1}\right) \cos \left[k \tilde{n}_{j} \cos \gamma_{j}\left(z-z_{j-1}\right)\right] .
\end{aligned}
$$

In order to calculate the field distribution in accordance with these formulas, it is necessary to express $u\left(z_{j-1}\right), v\left(z_{j-1}\right)$ through the amplitude of the incident wave. Let us denote as $E$ the tangential component of the electric vector of the incident wave in $S$ - or $P$-cases. Then for the transmitted wave $u(0)=t E, v(0)=q_{s} u(0)=q_{s} t E$. Calculating the field with the help of characteristic matrices of the layers onto the boundary $z=z_{j-1}$, we will obtain the sought for expression:

$$
\binom{u\left(z_{j-1}\right)}{v\left(z_{j-1}\right)}=M_{j-1} \ldots M_{1}\binom{1}{q_{s}} t E .
$$

### 1.2.2 Description of optical coating and presentation of calculation results

The geometrical thicknesses of the layers and the wavelength $\lambda$ enter the calculation formulas for $r$ and $t$ only in combination $k d=2 \pi d / \lambda$. This means that a simultaneous proportional change of the thicknesses of all layers and of the wavelength do not change the values of $r$ and $t$ (providing the refractive indices of all layers remain unchanged). Thus, in a spectral band where the dispersion of the refractive indices can be neglected, proportional changes of the thicknesses of all layers result in a shift of the reflectance and transmittance curves along the spectral band. In this connection, when describing the coating structure, it is convenient to use relative values of layer thicknesses rather than their absolute values.

Most widely recognized is the way of description where the optical thicknesses of the layers $n_{j} d_{j}$ are designated as fractions of $\lambda_{0} / 4$, with $\lambda_{0}$ being a certain basic wavelength. Let, for instance, a coating be a two-component one, consisting of alternating layers featuring high $n_{H}$ and low $n_{L}$ refractive indices. Consider the example of writing down structure of such a coating:

$$
4 S H 0.5 L 2 H L .
$$

Number 4 on the extreme left shows the total number of layers in the given coating, the following letter $S$ stands for the substrate. Then there is a consecutive list of the optical thicknesses in the direction from the substrate to the outer space. The letters $H$ and $L$ refer to the layers having the refractive indices of $n_{H}$ and $n_{L}$, respectively. The numerical coefficients before them show the optical thicknesses of the layer as fractions of $\lambda_{0} / 4$. If a coefficient is absent, the corresponding layer has a quarter-wave optical thickness. Thus, the coating described above has the first layer featuring the refractive index $n_{H}$ and the optical thickness $0.25 \lambda_{0}$, the second one has the refractive index $n_{L}$ and the optical thickness $0.125 \lambda_{0}$, the third one features the refractive index $n_{H}$ and the optical thickness $0.5 \lambda_{0}$, while the fourth one has the refractive index $n_{L}$ and the optical thickness $0.25 \lambda_{0}$.

The structural description is accompanied by indication of refractive indices for the substrate and the layers, for instance: $n_{s}=1.52, n_{H}=2.00, n_{L}=1.45$. A lack of any mention of the outer space proves that the outer space is the air. If the outer space is presented by a different substance, its refractive index is specified.

If re-iterated groups of layers occur in a coating, the number of reiterations is usually revealed as the power index and the re-iterated group is put in brackets. For example the two descriptions below are equivalent:

9SHLHLHLHLH and $\quad 9 S(H L)^{4} H$
When a coating has layers of three or more different materials, additional designations are introduced: $M, A, B, \ldots$. The letter $M$ is usually used to designate the layers with the refractive index lying between $n_{H}$ and $n_{L}$.

The basic wavelength $\lambda_{0}$ is not necessarily specified in the description of the coating structure. The spectral curves of the transmittance and reflectance can be conveniently presented as a dependence of $\lambda / \lambda_{0}$ or $\lambda_{0} / \lambda$. Substituting a specific value of $\lambda_{0}$ we determine the thicknesses of the coating layers and the spectral band where it will be valid.

Let us consider a seventeen-layer dielectric narrow bandpass filter featuring the $17 S(H L)^{4} 2 H(L H)^{4}$ design as an example. The refractive indices of the substrate and the layers have the following values: $n_{s}=1.52, \tilde{n}_{H}=2.3-i 0.0002, \tilde{n}_{L}=1.35-i 0.0002$. Thus, the materials of the layers are low absorbing. The calculation results for transmittance, reflectance and absorptance are given in Table 1.2. The Table 1.2. provides both a relative scale of wavelengths $\lambda / \lambda_{0}$ and that in nanometers, corresponding to $\lambda_{0}=500 \mathrm{~nm}$.

| $\lambda, \mathrm{nm}$ | $\lambda / \lambda_{0}$ | $\mathrm{~T}, \%$ | $R, \%$ | $\mathrm{~A}, \%$ |
| :---: | :---: | :---: | :---: | :---: |
| 480 | 0.960 | 0.146 | 99.721 | 0.133 |
| 485 | 0.970 | 0.238 | 99.616 | 0.146 |
| 492 | 0.984 | 0.780 | 98.982 | 0.238 |
| 497 | 0.994 | 5.196 | 93.786 | 1.018 |
| 499 | 0.998 | 31.286 | 63.049 | 5.665 |
| 500 | 1.000 | 83.401 | 1.597 | 15.002 |
| 501 | 1.002 | 31.429 | 62.827 | 5.744 |
| 503 | 1.006 | 5.312 | 93.624 | 1.064 |
| 508 | 1.016 | 0.829 | 98.917 | 0.254 |
| 515 | 1.030 | 0.264 | 99.585 | 0.151 |
| 520 | 1.040 | 0.165 | 99.702 | 0.133 |

Table 1.2: Transmittance, reflectance and absorptance of the seventeen- layer narrow bandpass filter: $17 S(H L)^{4} 2 H(L H)^{4}$ with $n_{s}=1.52, \tilde{n}_{H}=2.3-i 0.0002, \tilde{n}_{L}=1.35-i 0.0002$

### 1.2.3 Recurrent formulas for the amplitude transmittance and reflectance

Let us consider another convenient method frequently employed in calculating the amplitude transmittance and reflectance.

Let us return to formulas (1.1.17) and introduce, by analogy, '"intermediate" transmission and reflection coefficients.

$$
\begin{align*}
t_{j+1,0} & =\frac{2 q_{j+1}}{q_{j+1} u\left(z_{j}, k\right)+v\left(z_{j}, k\right)},  \tag{1.2.15}\\
r_{j+1,0} & =\frac{q_{j+1} u\left(z_{j}, k\right)-v\left(z_{j}, k\right)}{q_{j+1} u\left(z_{j}, k\right)+v\left(z_{j}, k\right)} .
\end{align*}
$$

They can be regarded as transmission and reflection coefficients for the subsystem of the layers from 1st through $j$-th, provided the outer space has the refractive index $n_{j+1}$. It is
convenient to hold further that the refractive index with the 0 index coincides with the refractive index of the substrate:

$$
n_{0}=n_{s} .
$$

Introduction of orderly arranged index pair $j+1,0$ in amplitude coefficients designations is convenient because the pair simultaneously indicates the refractive indices of the framing media ( $n_{j+1}$ and $n_{0}=n_{s}$ ) and the direction of the wave propagation (from medium with number $j+1$ to the substrate). It will be natural to mark the outer space with number $m+1$ and to mark all the values pertaining to this space by the subscript $m+1$. For instance, the outer space refractive index $n_{a}$ will have another designation of $n_{m+1}$. Since in this case

$$
q_{m+1}=q_{a},
$$

expressions (1.2.15) for $t_{m+1,0}, r_{m+1,0}$ will coincide with (1.1.17) and thus the following equations will be valid for the amplitude transmittance and reflectance of the whole coating

$$
\begin{equation*}
t=t_{m+1,0}, \quad r=r_{m+1,0} . \tag{1.2.16}
\end{equation*}
$$

Assuming $j=0$ and taking account of the initial conditions for the $u$ and $v$ functions at the point $z_{0}=0$, we obtain

$$
\begin{equation*}
t_{1,0}=\frac{2 q_{1}}{q_{1}+q_{s}}, \quad r_{1,0}=\frac{q_{1}-q_{s}}{q_{1}+q_{s}} . \tag{1.2.17}
\end{equation*}
$$

Those are nothing else but the Fresnel coefficients for the boundary between the first layer and the substrate. Expressions (1.2.17) set the initial conditions for the recurrent calculation method. The method itself consists in consecutive recalculations of the $t_{j+1,0}, r_{j+1,0}$ values from $j=1$ to $j=m$ in accordance with the recurrent formulas connecting these coefficients with $t_{j, 0}, r_{j, 0}$.

In order to obtain these recurrent formulas, let us write out the expressions for $t_{j, 0}, r_{j, 0}$ similar to (1.2.15):

$$
\begin{aligned}
t_{j, 0} & =\frac{2 q_{j}}{q_{j} u\left(z_{j-1}, k\right)+v\left(z_{j-1}, k\right)}, \\
r_{j, 0} & =\frac{q_{j} u\left(z_{j-1}, k\right)-v\left(z_{j-1}, k\right)}{q_{j} u\left(z_{j-1}, k\right)+v\left(z_{j-1}, k\right)} .
\end{aligned}
$$

From these formulas we obtain that

$$
u\left(z_{j-1}, k\right)=\frac{1+r_{j, 0}}{t_{j, 0}}, \quad v\left(z_{j-1}, k\right)=q_{j} \frac{1-r_{j, 0}}{t_{j, 0}} .
$$

Equations (1.2.7) can now be rewritten as

$$
\begin{aligned}
& u\left(z_{j}, k\right)=\frac{1+r_{j, 0}}{t_{j, 0}} \cos \varphi_{j}+i \frac{1-r_{j, 0}}{t_{j, 0}} \sin \varphi_{j}=\frac{1}{t_{j, 0}} e^{i \varphi_{j}}+\frac{r_{j, 0}}{t_{j, 0}} e^{-i \varphi_{j}} \\
& v\left(z_{j}, k\right)=i q_{j} \frac{1+r_{j, 0}}{t_{j, 0}} \sin \varphi_{j}+q_{j} \frac{1-r_{j, 0}}{t_{j, 0}} \cos \varphi_{j}=\frac{q_{j}}{t_{j, 0}} e^{i \varphi_{j}}-\frac{q_{j} r_{j, 0}}{t_{j, 0}} e^{-i \varphi_{j}}
\end{aligned}
$$

Substituting these expressions into (1.2.15), we will obtain the sought for recurrent formulas. Introducing the Fresnel transmission and reflection coefficients for the boundary between the $(j+1)$-st and the $j$-ih layer for the wave incident from the $(j+1)$-st layer

$$
t_{j+1, j}=\frac{2 q_{j+1}}{q_{j+1}+q_{j}}, \quad r_{j+1, j}=\frac{q_{j+1}-q_{j}}{q_{j+1}+q_{j}} .
$$

we will write down the recurrent formulas in the following way:

$$
\begin{align*}
t_{j+1,0} & =\frac{t_{j+1, j} t_{j, 0}}{e^{i \varphi_{j}}+r_{j+1, j} r_{j, 0} e^{-i \varphi_{j}}}, \\
r_{j+1,0} & =\frac{r_{j+1, j} e^{i \varphi_{j}}+r_{j, 0} e^{-i \varphi_{j}}}{e^{i \varphi_{j}}+r_{j+1, j} r_{j, 0} e^{-i \varphi_{j}}} . \tag{1.2.18}
\end{align*}
$$

Making recalculations with these formulas employed $m$ times for $j=1,2, \ldots, m$, we will determine the values of $t_{m+1,0}, r_{m+1,0}$, i.e., in accordance with (1.2.16) we will determine the amplitude transmittance and reflectance of the whole coating.

Note, that the obtained formulas are valid for both polarization cases. The type of polarization will only determine specific numerical values of the Fresnel coefficients in formulas (1.2.18) and the initial conditions (1.2.17). For the $S$ - and $P$-cases they are calculated applying the expressions cited in Table 1.1.

Transmittance and reflectance of the coating as a whole are obtained from the formulas

$$
T=\frac{q_{s}}{q_{a}}\left|t_{m+1,0}\right|^{2}, \quad R=\left|r_{m+1,0}\right|^{2}
$$

### 1.2.4 Approximate formulas

Approximate formulas for calculating transmittance and reflectance of a multilayer coating are of non-recurrent nature. Due to this fact, they prove to be convenient in a number of cases for analyzing the optical properties of the coating. Besides, they economize the calculation time needed to determine transmittance and reflectance, which can be quite essential for coating design. The precision of the approximate formulas, as is seen from the example below, is sufficiently high for many cases. Here we supply but final results. For the absorbing $m$-layer coating the following approximate formula is valid (Delano and Pegis, 1969):

$$
\frac{r}{t} \approx \frac{B}{D},
$$

where

$$
\begin{array}{ll}
B=r_{m+1, m}+\sum_{j=1}^{m} r_{j, j-1} \exp \left(-i \beta_{j}\right), \quad \beta_{j}=2 \sum_{l=j}^{m} \varphi_{l}, \\
D=t_{m+1, m} \prod_{j=1}^{m} t_{j, j-1} \exp \left(-i \varphi_{j}\right) .
\end{array}
$$

All the designations here have the same sense as in the previous subsection.
For a non-absorbing coating, reflectance and transmittance may be approximately calculated from the formulas, provided that there is no total internal reflection in any layer:

$$
R \approx \frac{|B|^{2}}{|B|^{2}+C}, \quad \frac{R}{T} \approx \frac{|B|^{2}}{C},
$$

where

$$
\begin{aligned}
|B|^{2} & =\left(r_{m+1, m}+\sum_{j=1}^{m} r_{j, j-1} \cos \beta_{j}\right)^{2}+\left(\sum_{j=1}^{m} r_{j, j-1} \sin \beta_{j}\right)^{2}, \\
C & =\prod_{j=1}^{m}\left(1-r_{j, j-1}^{2}\right) .
\end{aligned}
$$

As the reflectance of the coating grows, the accuracy of the formulas decreases. However, in many cases, the results of approximate calculations agree well with precise ones, up to the value of $R \approx 0.8$. Fig. 1.4 supplies the calculation results in accordance with the precise and approximate formulas for a three-layer coating featuring the structure 3 SHLH $, n_{s}=1.52, n_{H}=2.30, n_{L}=1.38$ at the normal angle of incidence. Some special types of coatings, such as multilayer dielectric mirrors, dielectric narrow bandpass filters may require and
get other approximate formulas, describing their major spectral properties with a sufficient degree of accuracy. They will be considered later in this Chapter.


Fig. 1.4: Comparative calculation of the reflectance of the three-layer 3SHLH coating following explicit and approximate formulas: 1 - explicit formulas, 2 - approximate formulas.

### 1.2.5 Other recurrent methods

Alongside with the above, other recurrent methods for calculating transmittance and reflectance of multilayer coatings prove convenient. Let us consider here the impedance and the admittance methods.

Let us introduce impedance both in the $S$ - and the $P$-cases through an equation

$$
Z(z, k)=\frac{u(z, k)}{v(z, k)}
$$

Thus, impedance is a ratio of the tangential components of the electric and the magnetic vectors.

Admittance is a value reverse to impedance:

$$
A(z, k)=\frac{1}{Z(z, k)} .
$$

Let us note that continuity of the tangential components of the electric and magnetic vector prescribes presupposed impedance and admittance continuity at the boundaries of the layers.

Impedance and admittance at the point $z_{a}$, i.e., at the outer boundary are known as the input impedance and input admittance, respectively. It follows from (1.1.17) that the amplitude reflectance can be simply expressed through the input impedance and admittance:

$$
\begin{equation*}
r(k)=\frac{q_{a} Z\left(z_{a}, k\right)-1}{q_{a} Z\left(z_{a}, k\right)+1}=\frac{q_{a}-A\left(z_{a}, k\right)}{q_{a}+A\left(z_{a}, k\right)} \tag{1.2.19}
\end{equation*}
$$

It is exactly this that provides a faculty to employ impedance and admittance in calculations.

Differential equations for impedance and admittance can be easily obtained from systems (1.1.9), (1.1.14) for the $u$ and $v$ functions. For impedance, they take the form of in the $S$-case

$$
\begin{equation*}
\frac{d Z}{d z}=i k\left\{1-\left[\tilde{\varepsilon}(z)-\alpha^{2}\right] Z^{2}\right\}, \tag{1.2.20}
\end{equation*}
$$

in the P-case

$$
\begin{equation*}
\frac{d Z}{d z}=i k\left\{1-\frac{\alpha^{2}}{\tilde{\varepsilon}(z)}-\tilde{\varepsilon}(z) Z^{2}\right\} \tag{1.2.21}
\end{equation*}
$$

The initial conditions for equations (1.2.20), (1.2.21) follow from (1.1.16):

$$
\begin{equation*}
Z(0, k)=\frac{1}{q_{s}} . \tag{1.2.22}
\end{equation*}
$$

Having found the solution of the equations (1.2.20), (1.2.21) with the initial conditions (1.2.22) ( $q_{s}$ are different for the $S$ - and $P$-cases), we can calculate the amplitude coefficients $r_{s, p}(k)$ from the formula (1.2.19).

For admittance we obtain the following equations:
in the $S$-case

$$
\begin{equation*}
\frac{d A}{d z}=i k\left\{\tilde{\varepsilon}(z)-\alpha^{2}-A^{2}\right\} \tag{1.2.23}
\end{equation*}
$$

in the $P$-case

$$
\begin{equation*}
\frac{d A}{d z}=i k\left\{\tilde{\varepsilon}(z)-\left[1-\frac{\alpha^{2}}{\tilde{\varepsilon}(z)}\right] A^{2}\right\} . \tag{1.2.24}
\end{equation*}
$$

The initial conditions for equations (1.2.23), (1.2.24) take the form of

$$
\begin{equation*}
A(0, k)=q_{s} . \tag{1.2.25}
\end{equation*}
$$

In the case of a multilayer coating, consisting of homogeneous layers, recurrent formulas for impedance and admittance are obtained directly from the formulas of the Abeles matrix method. Let us designate $Z_{j}$ and $A_{j}$ impedance and admittance at the boundary of the $j$-th and $(j+1)$-st layers. Dividing the equations (1.2.7) by each other, we obtain

$$
\begin{align*}
Z_{j} & =\frac{Z_{j-1} \cos \varphi_{j}+\left(i / q_{j}\right) \sin \varphi_{j}}{i q_{j} Z_{j-1} \sin \varphi_{j}+\cos \varphi_{j}},  \tag{1.2.26}\\
A_{j} & =\frac{i q_{j} \sin \varphi_{j}+A_{j-1} \cos \varphi_{j}}{\cos \varphi_{j}+\left(i / q_{j}\right) A_{j-1} \sin \varphi_{j}} \tag{1.2.27}
\end{align*}
$$

The initial values for equations (1.2.26), (1.2.27) are set by equations (1.2.22), (1.2.25):

$$
Z_{0}=\frac{1}{q_{s}}, \quad A_{0}=q_{s} .
$$

The values $Z_{m}$, $A_{m}$ obtained at the $m$-th step of the recurrent procedure reveal the input impedance and input admittance, and, thus, amplitude transmittance and reflectance for both cases of polarization are calculated from formula (1.2.19).

Note that by using values obtained in the course of recurrent procedure the amplitude transmittance can be obtained too. Let us see how it can be made employing the impedance method.

For this purpose, the electromagnetic field in every layer and in the outer space ought to be represented as the sum of the fields of two waves, one of which is transmitted in the positive direction of the $z$-axis, and the other in the negative one. Let us designate the complex amplitudes of the tangential components of the first one as $u^{+}, v^{+}$, and $u^{-}, v^{-}$of the second one. These amplitudes in the $j$-th layer are connected through equations:

$$
\begin{equation*}
v^{+}=-q_{j} u^{+}, \quad v^{-}=q_{j} u^{-} . \tag{1.2.28}
\end{equation*}
$$

The amplitude transmittance, as follows from the definition, is equal to

$$
\begin{equation*}
t(k)=\frac{u^{-}\left(z_{0}-0, k\right)}{u^{-}\left(z_{m}+0, k\right)} . \tag{1.2.29}
\end{equation*}
$$

We will obtain a recurrent formula for recalculating $u^{-}$field components from the left boundary of a multilayer coating to the right one. As follows from (1.2.28), $v=v^{+}+v^{-}=q_{j}\left(u^{-}-u^{+}\right)$. Using the equality $u^{-}+u^{+}=u$ we find that in the $j$-th layer

$$
\begin{equation*}
u^{-}=\frac{1}{2}\left(u+\frac{v}{q_{j}}\right) \tag{1.2.30}
\end{equation*}
$$

The $u^{-}$field component is not continuous at the boundaries of the layers. At the right boundary of the $j$-ih layer, that is when $z=z_{j}-0$ in accordance with (1.2.30) and continuity of the $u$ and $v$ functions

$$
u^{-}\left(z_{j}-0, k\right)=\frac{1}{2} v\left(z_{j}, k\right)\left(Z_{j}+\frac{1}{q_{j}}\right) .
$$

Similarly, at the left boundary of the $(j+1)$-st layer, that is at the point $z=z_{j}+0$

$$
u^{-}\left(z_{j}+0, k\right)=\frac{1}{2} v\left(z_{j}, k\right)\left(Z_{j}+\frac{1}{q_{j+1}}\right) .
$$

From these two equations we find how the $u^{-}$field component changes when crossing the boundary of the $j$-th and the $(j+1)$-st layer:

$$
\begin{equation*}
u^{-}\left(z_{j}+0, k\right)=u^{-}\left(z_{j}-0, k\right) \frac{Z_{j}+1 / q_{j+1}}{Z_{j}+1 / q_{j}} . \tag{1.2.31}
\end{equation*}
$$

In the $j$-th layer, a wave transmitting in the negative direction of the $z$-axis takes a dependence on $z$ in the form of $\exp \left(i k \tilde{n}_{j} \cos \gamma_{j} z\right)$. So the following equation is valid in the $j$-th layer

$$
\begin{equation*}
u^{-}\left(z_{j}-0, k\right)=u^{-}\left(z_{j-1}+0, k\right) e^{i \varphi_{j}} \tag{1.2.32}
\end{equation*}
$$

Applying alternatively formulas (1.2.31) and (1.2.32), we can express $\left.u^{-}\left(z_{m}+0, k\right)\right)$ through $u^{-}\left(z_{0}, k\right)$. Substituting the result into (1.2.29), we obtain

$$
\begin{equation*}
t=\frac{2 q_{1}}{q_{s}+q_{1}} \prod_{j=1}^{m} \frac{Z_{j}+1 / q_{j}}{Z_{j}+1 / q_{j+1}} e^{-i \varphi_{j}} . \tag{1.2.33}
\end{equation*}
$$

From (1.2.33) a similar expression for the amplitude transmittance through admittance can be easily obtained:

$$
\begin{equation*}
t=\frac{2 q_{a}}{q_{s}+q_{1}} \prod_{j=1}^{m} \frac{A_{j}+q_{j}}{A_{j}+q_{j+1}} e^{-i \varphi_{j}} . \tag{1.2.34}
\end{equation*}
$$

### 1.2.6 The admittance phase plane

Visual geometrical representations of the employed algorithms often help essentially in solving problems, contributing to a more thorough understanding of a problem in question. In the field of multilayer optics such representation can be achieved through a geometrical interpretation of the admittance calculation method. As was mentioned in the previous subsection, a convenience of employing admittance is due to the simplicity of the formula connecting the amplitude reflectance with the input admittance:

$$
r(k)=\frac{q_{a}-A\left(z_{a}, k\right)}{q_{a}+A\left(z_{a}, k\right)} .
$$

This connection is reversible: $A\left(z_{a}, k\right)$ is, in its turn, expressed through $r(k)$ according to the formula

$$
\begin{equation*}
A\left(z_{a}, k\right)=q_{a} \frac{1-r(k)}{1+r(k)} \tag{1.2.35}
\end{equation*}
$$

So design problems connected with the amplitude reflectance can be transformed for the input admittance. For instance, obtaining a multilayer coating with a given amplitude reflectance $r(k)$ is equivalent to obtaining a coating with a given input admittance determined through $r(k)$ according to the formula (1.2.35).


Fig. 1.5: Admittance phase plane
Admittance is a complex value. Geometrically it can be represented as a point on a complex plane (see Fig. 1.5). Let us call this plane an admittance phase plane. Assume, that a multilayer coating is non-absorbing. To make calculations and transformations brief, let us consider a normal incidence of light. As seen from the following, all the results are also valid for oblique wave incidence.

In case of a normal incidence, the formulas of the admittance method (see (1.2.25), (1.2.27) and (1.2.19)) take the following form:

$$
\begin{gather*}
A_{0} \equiv A(0, k)=n_{s},  \tag{1.2.36}\\
A_{j}=\frac{i n_{j} \sin \varphi_{j}+A_{j-1} \cos \varphi_{j}}{\cos \varphi_{j}+\left(i / n_{j}\right) A_{j-1} \sin \varphi_{j}} \tag{1.2.37}
\end{gather*}
$$

where $\varphi_{j}=k n_{j} d_{j}, j=1, \ldots, m$,

$$
\begin{equation*}
r(k)=\frac{n_{a}-A\left(z_{a}, k\right)}{n_{a}+A\left(z_{a}, k\right)} . \tag{1.2.38}
\end{equation*}
$$

where $A\left(z_{a}, k\right) \equiv A_{m}$.
Recurrent formulas (1.2.27) were obtained from formulas (1.2.7) of the matrix method. However, they could be obtained directly from the differential equation for admittance. In case of normal incidence, equations (1.2.23) and (1.2.24) coincide and take the form

$$
\begin{equation*}
\frac{d A}{d z}=i k\left[n^{2}(z)-A^{2}\right] \tag{1.2.39}
\end{equation*}
$$

For a multilayer coating the $n(z)$ function takes constant values inside every layer and features discontinuity at the boundaries of layers (see Fig 1.3). In compliance with the condition, the solution of equation (1.2.39) is continuous at the discontinuity points of $n(z)$. Let us consider the solution of this equation inside an arbitrary homogeneous layer. Let us designate its left boundary as $\hat{z}$, the value of admittance at this point as $\hat{A}$ (thus we set the initial condition for equation (1.2.39)) and the refractive index of the layer as $n$. Substituting into equation (1.2.39) one can easily check that the solution in the layer with the constant $n$ takes the form

$$
\begin{equation*}
A(z)=\frac{i n \sin k n(z-\hat{z})+\hat{A} \cos k n(z-\hat{z})}{\cos k n(z-\hat{z})+\left(i / n_{j}\right) \hat{A} \sin k n(z-\hat{z})} . \tag{1.2.40}
\end{equation*}
$$

Assuming $z=\hat{z}$ in (1.2.40), we see that $A(\hat{z})=\hat{A}$.
The solution in the form of (1.2.40) is valid, in particular, in the $j$-th layer. Here $n=n_{j}, \hat{z}=z_{j-1}, \hat{A}=A_{j-1}$, and if the right boundary $z_{j}$ is taken as $z$, then we obtain from (1.2.40) the recurrent formula (1.2.37).

We, however, need expression (1.2.40) for other purposes. As was mentioned earlier, admittance can be represented as a point on the complex plane. With $z$ changing, the value of admittance will change and, consequently, the point will plot some trajectory in the admittance phase ${ }^{3}$ plane.

Let us find out what the solution trajectory looks like. For this purpose, let us divide both the numerator and the denominator in (1.2.40) by $i \sin k n(z-\hat{z})$.
We will obtain

$$
\begin{equation*}
A(z)=\frac{n-i \hat{A} \cot k n(z-\hat{z})}{-i \cot k n(z-\hat{z})+\hat{A} / n} \tag{1.2.41}
\end{equation*}
$$

Let us introduce a new variable

$$
\xi=-\cot k n(z-\hat{z})
$$

and rewrite (1.2.41) as

[^2]\[

$$
\begin{equation*}
A=\frac{n+i \hat{A} \xi}{i \xi+\hat{A} / n} . \tag{1.2.42}
\end{equation*}
$$

\]

The function of $\xi$ in the right part of (1.2.42) is known as linear fractional. The relation (1.2.42) is conveniently regarded as a linear fractional transformation of the complex variable $\xi$ in the complex variable $A$.

With z changing from $\hat{z}$ to $\hat{z}+\pi /(k n)$ the variable $\xi$ runs along the real axis from $-\infty$ to $+\infty$. With further growth of $z$, the variable $\xi$ runs along the same values again. In accordance with the well-known property of the linear fractional function, circles in the $\xi$ complex plane transform into circles in the $A$ complex plane. A particular case of a circle is a straight line (a circle with an infinite radius). It follows, that with the change of the variable $z$ the variable $A(z)$ will move along some circle in the admittance phase plane (see Fig. 1.5).

Point $\hat{A}$ belongs to this circle. It corresponds to the value $z=\hat{z}$ or $\xi=-\infty$. With $z$ changing from $\hat{z}$ to $\hat{z}+\pi /(k n)$ (correspondingly, $\xi$ from $-\infty$ to $+\infty$ ) the point $A(z)$ makes a complete revolution.

Let us determine coordinates of the center and the radius of this circle. For this purpose we make use of the symmetrical property of the linear fractional transformation. Let us take two points $\xi=i$ and $\xi=-i$. They are symmetrical with respect to the real axis in the $\xi$ complex plane. In the course of the linear fractional transformation, points symmetrical with respect to the circle in $\xi$ plane are transformed to the points symmetrical with respect to the circle image in the $A$ plane. With $\xi= \pm i$ we obtain the following values of admittance from (1.2.42)

$$
A(z)=\frac{n-\hat{A}}{-1+\hat{A} / n}=-n, \quad A(z)=\frac{n+\hat{A}}{1+\hat{A} / n}=n .
$$

Thus, points $-n$ and $n$ are symmetrical with respect to our circle. Symmetrical points lie on the direct line passing through the center of the circle. It follows from here that the center of the circle lies on the real axis in the admittance phase plane.

Let us designate the circle center as $(0, \zeta)$ and its radius as $R$. By the very definition of symmetrical points they satisfy condition

$$
\begin{equation*}
(\zeta-n)(\zeta+n)=R^{2} \tag{1.2.43}
\end{equation*}
$$

This equation contains two unknown values: $\zeta$ and $R$. Taking into account that initial admittance value belongs to the circle in question, we can obtain another equation for their definition:

$$
\begin{equation*}
|\hat{A}-\zeta|^{2}=R^{2} \tag{1.2.44}
\end{equation*}
$$

Assuming (1.2.43) equal to (1.2.44), we obtain equation for determining the coordinate of the circle center:

$$
\zeta^{2}-n^{2}=|\hat{A}|^{2}-2 \operatorname{Re} \hat{A} \zeta+\zeta^{2}
$$

This gives

$$
\begin{equation*}
\zeta=\frac{|\hat{A}|^{2}+n^{2}}{2 \operatorname{Re} \hat{A}} \tag{1.2.45}
\end{equation*}
$$

Substituting (1.2.45) into (1.2.44), we obtain the radius of the circle. It is not complicated to check that the expression for the radius can be transformed into

$$
\begin{equation*}
R=\frac{\left|\hat{A}^{2}-n^{2}\right|}{2 \operatorname{Re} \hat{A}} \tag{1.2.46}
\end{equation*}
$$

As we see from (1.2.45), (1.2.46), the parameters of the circle along which the admittance for a given layer is moving, are completely determined by the refractive index of the layer and the admittance value on its left boundary. When passing to another layer, the admittance moves along a new circle with other parameters.

Let us show that all circles are located in the right part of the admittance phase plane. In the first layer the initial condition for the admittance is set by the equation (1.2.36). It follows from (1.2.46) that the center of this circle is located on the positive part of the real axis. According to (1.2.43),

$$
\begin{equation*}
R^{2}=\zeta^{2}-n^{2}<\zeta^{2}, \tag{1.2.47}
\end{equation*}
$$

and, consequently, the circle is located in the right part of the admittance phase plane (see Fig 1.5). So, the condition $\operatorname{Re} \hat{A}>0$ is valid for all the points on the circle. One of these points sets the initial condition for the second layer. So, the center of the second circle also lies in the positive part of the real axis. According to (1.2.47), any circle with the center $\zeta>0$ is completely located in the right half-plane. Developing this consideration, we find that the condition

$$
\begin{equation*}
\operatorname{Re} A>0 \tag{1.2.48}
\end{equation*}
$$

is always valid for admittance. It is the principal condition of the physical feasibility of admittance which could also be obtained directly from (1.2.23), (1.2.24). It is not difficult to show that it is completely equivalent to the condition of the physical feasibility of the amplitude reflectance $|r|<1$.

Note that one of the two points, $-n$ or $n$, symmetrical with respect to the circle, inevitably lies within the circle. Since the circle lies in the right half-plane, the inner point is evidently $n$.

Let us find now in what direction the admittance movement along the circle in the phase plane is effected.

The linear fractional transformation is a particular case of a conformal transformation. In accordance with the well known property, the direction of passing round the domain in a complex plane remains unchanged while the conformal transformation is done. The point $-i$ lies in the $\xi$ plane to the right of the real axis with respect to the variable $\xi$ passing from $-\infty$ to $+\infty$. So, its image, the point n during passing round the circle in the $A$-plane ought to remain to the right side of this circle. It follows from here that the movement along the circle in the admittance phase plane always goes clockwise.

As was shown above, the center and the radius of the circle are determined by the layer refractive index and the admittance initial value on its left boundary. With different initial values, the circles along which point A moves are different. Note, however, that in accordance with (1.2.43), the coordinates of the centers and radii of all these circles are unambiguously connected between themselves:

$$
\zeta^{2}-n^{2}=R^{2} .
$$

Thus, for a given layer with the refractive index $n$ there is an infinite set of circles which can serve as admittance movement paths. Fig. 1.5 plots in dashed lines a number of circles of this set. With $R \rightarrow 0$ they urge towards the point $n$.

For layers featuring different refractive indices, the set of circles is different. On the whole, each one of these sets completely fills the right side of the half-plane. Let us also note, that all the circles of the set are inserted into each other and do not cross.

We can consider the problem of eliminating the reflectance from the boundary between the substrate and the outer space as an example of a practical application of the discussed geometrical representations. According to the formula (1.2.38), it is necessary and sufficient for complete eliminating the reflectance to have the input admittance equal to the refractive index of
the outer space. Geometrically, it means that the admittance trajectory ought to terminate at the point $n_{a}$ (Fig. 1.6). The beginning of the trajectory is point $n_{s}$. Assuming $A=n_{s}, n=n_{1}$ in (1.2.45), (1.2.46), we will find the parameters of the circle along which the admittance moves in the first layer:

$$
\zeta_{1}=\frac{n_{s}^{2}+n_{1}^{2}}{2 n_{s}}, \quad R_{1}=\frac{\left|n_{s}^{2}-n_{1}^{2}\right|}{2 n_{s}}
$$

This circle is marked as number 1 in Fig. 1.6 (representing a $n_{1}>n_{s}$ case). The set of circles corresponding to the layer featuring refractive index $n_{2}$, has a circle crossing the $n_{a}$ point. Let us show that its parameters can also be found from the formulas (1.2.45), (1.2.46). Actually, the condition of belonging to a set of circles takes the form (1.2.43) with $n=n_{2}$, and the condition of passing through point $n_{a}$ takes the form of (1.2.44) with $\hat{A}=n_{a}$. So, the relations (1.2.45), (1.2.46), obtained from (1.2.43), (1.2.44) are valid for the case under consideration (generally speaking, $\hat{A}$ can be any point of the circle). Thus, the center and the radius of the circle are determined from the formulas

$$
\zeta_{2}=\frac{n_{a}^{2}+n_{2}^{2}}{2 n_{a}}, \quad R_{2}=\frac{\left|n_{a}^{2}-n_{2}^{2}\right|}{2 n_{a}} .
$$

If the circle thus plotted (it is designated as number 2 in Fig. 1.6) crosses the first circle, then the admittance trajectory leading from point $n_{s}$ to point $n_{a}$ can be easily found. One of the two trajectories of this type is plotted in a thick solid line in Fig. 1.6. Thus, crossing of the circles 1 and 2 means a possibility of reduction reflection to zero with the help of a two-layer coating. The thicknesses of the layers are determined by the admittance value at the crossing point of the two circles.

In conclusion of this section, let us consider another important result which can be easily established on the basis of the analyzed geometrical interpretation of admittance.

Let us find out whether a multilayer coating with any pre-set amplitude reflectance (at a certain fixed wavelength) can be obtained. We showed earlier that the task of obtaining a predetermined amplitude reflectance is completely equivalent to the task of obtaining a corresponding input admittance. So, let $A_{a}$ be an arbitrarily set input admittance (naturally, meeting the requirements of a physical feasibility (1.2.48)). Let us consider a two-component multilayer coating consisting of layers with alternating refractive indices $n_{1}$ and $n_{2}$. Let us
designate the sets of circles for such layers as $C_{1}$ and $C_{2}$. In Fig. 1.7, the solid lines show the circles of the first set, and the dashed lines - those of the second one (assuming $n_{2}<n_{s}<n_{1}$ ). Circles of every set fill completely the whole of the right half-plane. Note, that any circle of one set crosses an infinite number of circles of the other set. Due to this, there is a possibility for the admittance trajectory to reach any $A_{a}$ point in the right half-plane. The initial stretch of the trajectory always goes along a circle from $C_{1}$ set designated as 1 in the Fig. 1.7. Depending on the thickness of the first layer, transition of the trajectory into one of the circles of the $C_{2}$ set can occur at any point of circle 1 . It is evident from the picture, that the circle designated as 2 , confines the area into which the admittance trajectory of a two-layer system can get. Similarly, circle 3 of the $C_{1}$ set limits the area of admittance values which can be obtained through a threelayer system. Thus, increasing the number of layers, we can always obtain any sought value of the input admittance.


Fig. 1.6: Design of antireflection coating with the help of admittance phase plane.

The established result can be formulated as a theorem.
Theorem 1.1 There exits a multiplayer coating having any pre-set amplitude reflectance at any pre-set wavelength.


Fig. 1.7: Admittance trajectories of two-component multilayer coating

### 1.3 Some common properties of multilayer coatings

Multilayer optical coatings have a number of common properties whose appreciation proves useful in solving numerous applied problems of multilayer optics. Great is the contribution of P. G. Kard in studying these properties. The results of his research in this field have been summed up in his book published in 1971, which is of no less importance nowadays. Here we will consider only most essential properties from the point of view of applied optics. We will employ a sufficiently simple but rather a common approach in establishing these properties, which, in itself, is useful in solving a number of problems of multilayer optics. In this section formulas are deduced for transmittance and reflectance of two multilayer systems combination. They assist in investigating many essential types of multilayer coatings, such as multilayer interference filters, for example.

### 1.3.1 The relation between the characteristic matrix and spectral coefficients of a multilayer coating

In this subsection we will establish simple relations connecting the characteristic matrix with multilayer system amplitude transmittance and reflectance for waves passing in the straight and reverse direction (i.e., from the substrate to the outer space). These relations make the basis for almost all further results of the section.

We will make simultaneous deductions for both cases of polarization. As shown in the previous section, the characteristic matrix connects field components on the boundaries of a multilayer coating with the outer space and the substrate:

$$
\begin{equation*}
\binom{u\left(z_{a}, k\right)}{v\left(z_{a}, k\right)}=M\binom{u(0, k)}{v(0, k)} . \tag{1.3.1}
\end{equation*}
$$

If we set the initial conditions in the right part of (1.3.1)

$$
\begin{equation*}
u(0, k)=1, \quad v(0, k)=q_{s}, \tag{1.3.2}
\end{equation*}
$$

then we obtain in the left part of (1.3.1) the values which enter formulas (1.1.17) for the amplitude transmittance and reflectance:

$$
\begin{align*}
& t(k)=\frac{2 q_{a}}{q_{a} u\left(z_{a}, k\right)+v\left(z_{a}, k\right)},  \tag{1.3.3}\\
& r(k)=\frac{q_{a} u\left(z_{a}, k\right)-v\left(z_{a}, k\right)}{q_{a} u\left(z_{a}, k\right)+v\left(z_{a}, k\right)} .
\end{align*}
$$

Let us express $u\left(z_{a}, k\right), v\left(z_{a}, k\right)$ from (1.3.3) through the amplitude coefficients:

$$
\begin{equation*}
u\left(z_{a}, k\right)=\frac{1+r}{t}, \quad v\left(z_{a}, k\right)=\frac{q_{a}(1-r)}{t} . \tag{1.3.4}
\end{equation*}
$$

Substituting (1.3.2) and (1.3.3) into (1.3.1), we will obtain

$$
\begin{equation*}
\binom{\frac{1+r}{t}}{\frac{q_{a}(1-r)}{t}}=M\binom{1}{q_{s}} \tag{1.3.5}
\end{equation*}
$$

Let us now introduce the amplitude transmittance and reflectance for the wave incident from the substrate onto the multilayer coating. The angle of incidence is assumed to be equal to $\gamma_{s}$, that is the value of the transmittance angle at which the wave passes in the substrate when the light is incident from the outer space at angle $\gamma_{a}$.

For brevity, let us call these coefficients amplitude transmittance and reflectance in the reverse direction and designate them as $t^{\prime}$ and $r^{\prime}$.

The $t^{\prime}$ and $r^{\prime}$ coefficients can be found similarly to the way the formulas (1.1.17) were obtained in section 1.1. We should normalize the field with respect to the transmitted wave amplitude. Now it is a wave passing in the outer space at the $\gamma_{a}$ angle in the direction of the coordinate $z$ growth. For this wave $v=-q_{a} u$. So, the boundary field conditions will be set now at point $z=z_{a}$ in the following form:

$$
\begin{equation*}
u\left(z_{a}, k\right)=1, \quad v\left(z_{a}, k\right)=-q_{a} . \tag{1.3.6}
\end{equation*}
$$

If $u(0, k), v(0, k)$ are corresponding values of the field components at the boundary with the substrate, then the $t^{\prime}$ and $r^{\prime}$ coefficients will be expressed similarly to formulas (1.1.17):

$$
\begin{align*}
& t^{\prime}=\frac{2 q_{s}}{q_{s} u(0, k)-v(0, k)},  \tag{1.3.7}\\
& r^{\prime}=\frac{q_{s} u(0, k)+v(0, k)}{q_{s} u(0, k)-v(0, k)} .
\end{align*}
$$

Let us express $u(0, k), v(0, k)$ from (1.3.7) through the amplitude coefficients in the reverse direction:

$$
\begin{equation*}
u(0, k)=\frac{1+r^{\prime}}{t^{\prime}}, \quad v(0, k)=\frac{q_{s}\left(r^{\prime}-1\right)}{t^{\prime}} \tag{1.3.8}
\end{equation*}
$$

Substituting (1.3.6) and (1.3.8) into (1.3.1), we obtain:

$$
\begin{equation*}
\binom{1}{-q_{a}}=M\binom{\frac{1+r^{\prime}}{t^{\prime}}}{\frac{q_{s}\left(r^{\prime}-1\right)}{t^{\prime}}} \tag{1.3.9}
\end{equation*}
$$

It is convenient, for further use, to introduce two matrices:

$$
Q_{a}=\left(\begin{array}{cc}
1 & 1 \\
q_{a} & -q_{a}
\end{array}\right), \quad Q_{s}=\left(\begin{array}{cc}
1 & 1 \\
q_{s} & -q_{s}
\end{array}\right)
$$

With their help, equation ( 1.3.5) is written down as

$$
Q_{a}\binom{\frac{1}{t}}{\frac{r}{t}}=M Q_{s}\binom{1}{0}
$$

and equation (1.3.9) as

$$
Q_{a}\binom{0}{1}=M Q_{s}\binom{\frac{r^{\prime}}{t^{\prime}}}{\frac{1}{t^{\prime}}}
$$

Now we can write both these equations into one matrix equation:

$$
Q_{a}\left(\begin{array}{cc}
\frac{1}{t} & 0 \\
\frac{r}{t} & 1
\end{array}\right)=M Q_{s}\left(\begin{array}{cc}
1 & \frac{r^{\prime}}{t^{\prime}} \\
0 & \frac{1}{t^{\prime}}
\end{array}\right)
$$

Let us rewrite it in the following form:

$$
\left(\begin{array}{cc}
\frac{1}{t} & 0  \tag{1.3.10}\\
\frac{r}{t} & 1
\end{array}\right)\left(\begin{array}{cc}
1 & \frac{r^{\prime}}{t^{\prime}} \\
0 & \frac{1}{t^{\prime}}
\end{array}\right)^{-1}=Q_{a}^{-1} M Q_{s}
$$

Let us designate the left matrix product as $C$. Finding the inverse matrix and multiplying the matrices then, we obtain

$$
C=\left(\begin{array}{cc}
\frac{1}{t} & -\frac{r^{\prime}}{t}  \tag{1.3.11}\\
\frac{r}{t} & \frac{1}{t^{+}}
\end{array}\right)
$$

(1.3.11) where $t^{+}$stands for the value

$$
\begin{equation*}
t^{+}=\frac{t}{t t^{\prime}-r r^{\prime}} . \tag{1.3.12}
\end{equation*}
$$

In radiophysics, similar $C$ matrices are customarily called transfer matrices. We will stick to the term in the multilayer optics. It presents no special difficulty to analyze the physical meaning of the $C$-matrix. Assume that $E_{A}$-amplitude wave is incident from the outer space side. For the sake of simplicity, let us consider a normal incidence case. The amplitudes of the transmitted and reflected waves $E_{T}$ and $E_{T}$ are connected with $E_{A}$ by the equations

$$
E_{T}=t E_{A}, \quad E_{R}=r E_{A} .
$$

From here we obtain

$$
\begin{equation*}
E_{T}=\frac{1}{t} E_{T}, \quad E_{R}=\frac{r}{t} E_{T} . \tag{1.3.13}
\end{equation*}
$$

These equations are equivalent to the following matrix equation

$$
\begin{equation*}
\binom{E_{A}}{E_{R}}=C\binom{E_{T}}{0} \tag{1.3.14}
\end{equation*}
$$

Let now an $E_{A}^{\prime}$ amplitude wave is simultaneously incident from the substrate side, and $E_{T}^{\prime}$ and $E_{R}^{\prime}$ are corresponding amplitudes of the transmitted and reflected waves (see Fig. 1.8). Similarly to (1.3.13)

$$
\begin{equation*}
E_{A}^{\prime}=\frac{1}{t^{\prime}} E_{T}^{\prime}, \quad E_{R}^{\prime}=\frac{r^{\prime}}{t^{\prime}} E_{T}^{\prime} . \tag{1.3.15}
\end{equation*}
$$

Let us find the matrix inverse to $C$-matrix. It is equal to

$$
C^{-1}=\left(\begin{array}{cc}
\frac{t t^{\prime}-r r^{\prime}}{t^{\prime}} & \frac{r^{\prime}}{t^{\prime}} \\
-\frac{r}{t^{\prime}} & \frac{1}{t^{\prime}}
\end{array}\right) .
$$

Applying it, equations (1.3.15) can be written down as

$$
\begin{equation*}
\binom{E_{R}^{\prime}}{E_{A}^{\prime}}=C^{-1}\binom{0}{E_{T}^{\prime}} . \tag{1.3.16}
\end{equation*}
$$

Multiplying both parts of (1.3.16) by the $C$-matrix, we obtain

$$
\begin{equation*}
\binom{0}{E_{T}^{\prime}}=C\binom{E_{R}^{\prime}}{E_{A}^{\prime}} . \tag{1.3.17}
\end{equation*}
$$

Due to the superposition principle, the fields considered above, add up. Analytically, it is equivalent to the addition of equations (1.3.14) and (1.3.17). As a result, we obtain

$$
\begin{equation*}
\binom{E_{A}}{E_{R}+E_{T}^{\prime}}=C\binom{E_{R}^{\prime}+E_{T}}{E_{A}^{\prime}} \tag{1.3.18}
\end{equation*}
$$



Figure 1.8: Determination of transfer matrix

Let us note that $E_{A}$ and $E_{R}+E_{T}^{\prime}$ are amplitudes of the waves passing in the outer space in the negative and the positive directions of the $z$-axis, respectively, and $E_{R}^{\prime}+E_{T}$ and $E_{A}^{\prime}$ are amplitudes of similar waves in the substrate. Equation (1.3.18) establishes a connection between these amplitudes. Thus, the $C$-matrix describes the transference properties of the multilayer coating in the discussed sense.

Equation (1.3.10) is written down with the help of the transfer matrix in the form

$$
\begin{equation*}
C=Q_{a}^{-1} M Q_{s} \tag{1.3.19}
\end{equation*}
$$

Hence, we obtain the expression for the characteristic matrix of the multilayer coating through its transfer matrix, and, thus, through the amplitude transmittance and reflectance coefficients for the direct and reverse waves:

$$
\begin{equation*}
M=Q_{a} C Q_{s}^{-1} . \tag{1.3.20}
\end{equation*}
$$

### 1.3.2 Relations between spectral characteristics for the direct and reverse waves

Let us note, first of all, that irrespective of the fact whether the layer is absorbing or nonabsorbing, the determinant of its characteristic matrix (see (1.2.9)) is always equal to 1 . So this is also valid for the determinant of the characteristic matrix of the multilayer coating:

$$
\begin{equation*}
\operatorname{det} M=1 \tag{1.3.21}
\end{equation*}
$$

It follows from (1.3.19) due to (1.3.21) that

$$
\begin{equation*}
\operatorname{det} C=\operatorname{det} Q_{a}^{-1} \operatorname{det} Q_{s}=\frac{\operatorname{det} Q_{s}}{\operatorname{det} Q_{a}} . \tag{1.3.22}
\end{equation*}
$$

Calculating the determinants entering the equations (1.3.22), we obtain

$$
\begin{equation*}
\frac{t^{\prime}}{t}=\frac{q_{s}}{q_{a}} . \tag{1.3.23}
\end{equation*}
$$

Let us further assume that both the substrate and the outer space are non-absorbing. The transmittances for the waves passing directly and reversely are determined by the equations

$$
T=\frac{q_{s}}{q_{a}}|t|^{2}, \quad T^{\prime}=\frac{q_{a}}{q_{s}}\left|t^{\prime}\right|^{2} .
$$

It follows from (1.3.23) that $T=T^{\prime}$. Thus, the following property is revealed:
Property 1. Transmittances for the waves passing directly and reversely through a multilayer coating, coincide.

If all layers of a multilayer coating are non-absorbing, equality of the reflectances for the directly and reversely passing waves follows from Property $1: R=R^{\prime}$. However, this equation is not valid in the general case of the absorbing layered medium.

Of the energy equations

$$
T+R+A=1, \quad T^{\prime}+R^{\prime}+A^{\prime}=1 .
$$

and Property 1 only the following equality is valid

$$
R+A=R^{\prime}+A^{\prime} .
$$

As was shown by P.G.Kard (1971), absorbing multilayer coatings featuring significant differences between reflectances in direct and reverse directions may exist. This property can be employed to design asymmetric mirrors for reflecting interferometers (Troitski, 1985).

Amplitude transmittance and reflectance are complex values. Let us write them down as

$$
t=\tau e^{i \phi}, \quad r=\rho e^{i \Delta} .
$$

The values $\delta$ and $\Delta$ are known as phase changes on transmission and reflection. They depend on the wavenumber $k$ (or the wavelength $\lambda$ ) and are essential spectral characteristics of multilayer coatings.

Let us designate as $\delta^{\prime}$ and $\Delta^{\prime}$ the phase changes on transmission and reflection for the wave passing in the reverse direction.

Since the right part of (1.3.23) is a real number, it follows from the equation that

$$
\begin{equation*}
\delta=\delta^{\prime} . \tag{1.3.24}
\end{equation*}
$$

Let us now assume that a multilayer coating is non-absorbing. The phase thicknesses of each layer $\varphi_{j}=k n_{j} \cos \gamma_{j} d_{j}$ are real numbers, so elements of the principal diagonals of the characteristic matrices are real numbers, and the elements of the subsidiary diagonals are imaginary numbers. By multiplying these matrices, it is easy to see that the $m_{11}$ and $m_{22}$ elements of the characteristic matrix of the whole coating are real numbers, while $m_{12}$ and $m_{21}$ are imaginary numbers.

Calculating the product of the matrices in the right part of (1.3.19), we obtain for matrix $C$ :

$$
\begin{gathered}
C=\frac{1}{2}\left(\begin{array}{cc}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{array}\right), \\
C_{11}=\left(m_{11}+\frac{q_{s}}{q_{a}} m_{22}\right)+\left(q_{s} m_{12}+\frac{m_{21}}{q_{a}}\right), \\
C_{12}=\left(m_{11}-\frac{q_{s}}{q_{a}} m_{22}\right)-\left(q_{s} m_{12}-\frac{m_{21}}{q_{a}}\right), \\
C_{21}=\left(m_{11}-\frac{q_{s}}{q_{a}} m_{22}\right)+\left(q_{s} m_{12}-\frac{m_{21}}{q_{a}}\right), \\
C_{22}=\left(m_{11}+\frac{q_{s}}{q_{a}} m_{22}\right)-\left(q_{s} m_{12}+\frac{m_{21}}{q_{a}}\right) .
\end{gathered}
$$

Taking account of the above property of the characteristic matrix elements we see, that the elements on the principal diagonal of matrix $C$ are conjugate complex numbers. Elements on the subsidiary diagonal are also conjugate complex numbers. It means that

$$
\begin{equation*}
\frac{1}{t^{+}}=\frac{1}{t^{*}}, \quad-\frac{r^{\prime}}{t}=\frac{r^{*}}{t^{*}} . \tag{1.3.25}
\end{equation*}
$$

The second equation gives rise to the following relation between the phase changes on reflection in direct and reverse directions:

$$
\begin{equation*}
\Delta^{\prime}=\pi-\Delta+2 \delta \tag{1.3.26}
\end{equation*}
$$

Let us sum up the obtained results.
Property 2. In case of a non-absorbing multilayer coaling for phase changes on transmission and reflection, there exist relations (1.3.24) and (1.3.26). The relation (1.3.24) is valid with the absorbing layers, too.

### 1.3.3 Transmittance and reflectance of the combination of two multilayer subsystems

Assume, that a multilayer coating is obtained through uniting two subsystems of layers, designated as 1 and 2, with a spacer layer having thickness $d$ and refractive index $n$ (see Fig. 1.9). Let us introduce the amplitude transmittance and reflectance for the waves passing in direct (from the outer space to the substrate) and reverse (from the substrate to the outer space) directions. These coefficients will be, respectively, designated with 1 and 2 indices. For the sake of certainty, in setting the amplitude coefficients of subsystems 1 and 2 , we consider that subsystem 1 is framed by homogeneous media with refractive indices $n_{s}$ and $n$ and subsystem 2 is framed by homogeneous media with refractive indices $n$ and $n_{a}$. The $n$-refractive index medium for subsystem 1 plays the part of the outer space, and that of the substrate for subsystem 2. The wave angles in all media are determined by the Snell law:

$$
n_{s} \sin \gamma_{s}=n \sin \gamma=n_{a} \sin \gamma_{a} .
$$

Let us find expressions for the amplitude transmittance and reflectance of the multilayer coating through the amplitude coefficients of its subsystems.

Let $M_{1}$ and $M_{2}$ be characteristic matrices of subsystems 1 and 2, and $M_{p}$ is a characteristic matrix of the spacer layer. The product of these matrices gives characteristic matrix of the multilayer coating

$$
\begin{equation*}
M=M_{2} M_{p} M_{1} . \tag{1.3.27}
\end{equation*}
$$

Let us now express matrices $M, M_{2}, M_{1}$ through the corresponding transfer matrices. According to (1.3.20),

$$
M=Q_{a} C Q_{s}^{-1} .
$$

Similarly

$$
M_{2}=Q_{a} C_{2} Q^{-1}, \quad M_{1}=Q C_{1} Q_{s}^{-1} .
$$

where $C_{1,2}$ are transfer matrices of subsystems 1 and 2 , and Q is a matrix similar to $\mathrm{Q}_{s}$ and $\mathrm{Q}_{a}$ for the $n$-refractive index medium:

$$
\mathrm{Q}=\left(\begin{array}{cc}
1 & 1 \\
q & -q
\end{array}\right) .
$$

Here $q=n \cos \gamma$ in the $S$-case and $q=n / \cos \gamma$ in the $P$-case.


Figure 1.9: Multilayer coating obtained by combining two multilayer subsystems

Substituting expressions for matrices $M, M_{2}, M_{1}$ in (1.3.27), reducing both parts of the resulting equation by $\mathrm{Q}_{a}$ and $\mathrm{Q}_{s}^{-1}$ we have:

$$
\begin{equation*}
C=C_{2} Q^{-1} M_{p} Q C_{1} \tag{1.3.28}
\end{equation*}
$$

Let us write down explicitly the characteristic matrix of the spacer layer:

$$
M_{p}=\left(\begin{array}{cc}
\cos \varphi & \frac{i}{q} \sin \varphi \\
i q \sin \varphi & \cos \varphi
\end{array}\right)
$$

Here, $\varphi=k n \cos \gamma d$ is a phase thickness of the spacer layer.
The inverse matrix $\mathrm{Q}^{-1}$ is equal to:

$$
\mathrm{Q}^{-1}=\frac{1}{2}\left(\begin{array}{cc}
1 & \frac{1}{q} \\
1 & -\frac{1}{q}
\end{array}\right) .
$$

Multiplying the three middle matrices in (1.3.28) by each other, we obtain:

$$
\mathrm{Q}^{-1} M_{p} \mathrm{Q}=\left(\begin{array}{cc}
e^{i \varphi} & 0  \tag{1.3.29}\\
0 & e^{-i \varphi}
\end{array}\right)
$$

The transfer matrices $C_{1}$ and $C_{2}$ in (1.3.28) are expressed through the subsystem amplitude coefficients using similar (1.3.11) formulas:

$$
C_{1}=\left(\begin{array}{cc}
\frac{1}{t_{1}} & -\frac{r_{1}^{\prime}}{t_{1}} \\
\frac{r_{1}}{t_{1}} & \frac{1}{t_{1}^{+}}
\end{array}\right), \quad C_{2}=\left(\begin{array}{cc}
\frac{1}{t_{2}} & -\frac{r_{2}^{\prime}}{t_{2}} \\
\frac{r_{2}}{t_{2}} & \frac{1}{t_{2}^{+}}
\end{array}\right)
$$

where $t_{1}^{+}$and $t_{2}^{+}$look similar to (1.3.12). So, taking account of (1.3.29), the equation (1.3.28) is written down in the form:

$$
\left(\begin{array}{cc}
\frac{1}{t} & -\frac{r^{\prime}}{t}  \tag{1.3.30}\\
\frac{r}{t} & \frac{1}{t^{+}}
\end{array}\right)=\left(\begin{array}{cc}
\frac{1}{t_{2}} & -\frac{r_{2}^{\prime}}{t_{2}} \\
\frac{r_{2}}{t_{2}} & \frac{1}{t_{2}^{+}}
\end{array}\right)\left(\begin{array}{cc}
e^{i \varphi} & 0 \\
0 & e^{-i \varphi}
\end{array}\right)\left(\begin{array}{cc}
\frac{1}{t_{1}} & -\frac{r_{1}^{\prime}}{t_{1}} \\
\frac{r_{1}}{t_{1}} & \frac{1}{t_{1}^{+}}
\end{array}\right)
$$

In order to express the $t$ and $r$ amplitude coefficients, it is sufficient now to find the first column of the matrix product in the right part of (1.3.30). We will obtain then:

$$
\begin{aligned}
& \frac{1}{t}=\frac{e^{i \varphi}-r_{1} r_{2}^{\prime} e^{-i \varphi}}{t_{1} t_{2}} \\
& \frac{r}{t}=\frac{r_{2} e^{i \varphi}+r_{1}\left(t_{2} t_{2}^{\prime}-r_{2} r_{2}^{\prime}\right) e^{-i \varphi}}{t_{1} t_{2}}
\end{aligned}
$$

and get from here the final expressions for $t$ and $r$ :

$$
\begin{align*}
& t=\frac{t_{1} t_{2} e^{-i \varphi}}{1-r_{1} r_{2}^{\prime} e^{-2 i \varphi}},  \tag{1.3.31}\\
& r=\frac{r_{2}+r_{1}\left(t_{2} t_{2}^{\prime}-r_{2} r_{2}^{\prime}\right) e^{-2 i \varphi}}{1-r_{1} r_{2}^{\prime} e^{-2 i \varphi}} . \tag{1.3.32}
\end{align*}
$$

Note, that expressions (1.3.31), (1.3.32) are of the most general character. They are valid for both field components. Both multilayer subsystems and the spacer layer can be absorbing.

We can consider the combination of two multilayer subsystems without spacer layer between them. In the latter case, $\varphi$ is held equal to zero in (1.3.31) and (1.3.32).

Now we will obtain expressions for the energy transmittance and reflectance coefficients. Let us introduce the transmittances and reflectances for subsystems 1 and 2:

$$
T_{1}=\frac{q_{s}}{q}\left|t_{1}\right|^{2}, \quad R_{1}=\left|r_{1}\right|^{2}, \quad T_{2}=\frac{q}{q_{a}}\left|t_{2}\right|^{2}, \quad R_{2}=\left|r_{2}\right|^{2}
$$

From (1.3.31) we obtain

$$
\begin{equation*}
T=\frac{q_{s}}{q_{a}}|t|^{2}=\frac{q_{s}\left|t_{1}\right|^{2}\left|t_{2}\right|^{2}}{q_{a}\left|1-r_{1} r_{2}^{\prime} e^{-2 i \varphi}\right|^{2}}=\frac{T_{1} T_{2}}{\left|1-\sqrt{R_{1} R_{2}} e^{i\left(\Delta_{1}+\Delta_{2}^{\prime}-2 \varphi\right)}\right|^{2}} \tag{1.3.33}
\end{equation*}
$$

Here $\Delta_{1}$ and $\Delta_{2}^{\prime}$ are phase changes on reflection from subsystems 1 and 2 at the side of the spacer layer. Let us designate

$$
\theta=\frac{1}{2}\left(\Delta_{1}+\Delta_{2}^{\prime}-2 \varphi\right)
$$

It can be easily checked that the square module in the denominator in (1.3.33) is equal to $1+R_{1} R_{2}-2 \sqrt{R_{1} R_{2}} \cos 2 \theta$. So, the expression (1.3.33) for the transmittance of the coating is transformed into

$$
\begin{equation*}
T=\frac{T_{1} T_{2}}{1+R_{1} R_{2}-2 \sqrt{R_{1} R_{2}} \cos 2 \theta} \tag{1.3.34}
\end{equation*}
$$

In order to simplify the final expression for the reflectance of the coating, let us assume multilayer subsystem 2 non-absorbing. In this case, in accordance with the first equation in (1.3.25) and Property 2 (equation (1.3.26)),

$$
t_{1} t_{2}^{\prime}-r_{2} r_{2}^{\prime}=\frac{t_{2}}{t_{2}^{+}}=e^{2 i \delta}=-e^{i\left(\Delta_{2}^{\prime}+\Delta_{2}\right)} .
$$

So the numerator in (1.3.32) can be written down as

$$
e^{i \Delta_{2}}\left[\sqrt{R_{2}}-\sqrt{R_{1}} e^{\left(\Delta_{1}+\Delta_{2}^{\prime}-2 \varphi\right)}\right] .
$$

The square module of the value is equal to $R_{1}+R_{2}-2 \sqrt{R_{1} R_{2}} \cos 2 \theta$. Taking this into account, we obtain the following expression for $R$ :

$$
\begin{equation*}
R=\frac{R_{1}+R_{2}-2 \sqrt{R_{1} R_{2}} \cos 2 \theta}{1+R_{1} R_{2}-2 \sqrt{R_{1} R_{2}} \cos 2 \theta} \tag{1.3.35}
\end{equation*}
$$

Expressions (1.3.34), (1.3.35) form the basis for a design method of effective interfaces (see Smith, 1957). The expressions are also widely applied for analyzing important types of multilayer coatings, including mirrors and filters.

### 1.4 The multilayer coatings optical properties: impact of variations in the layer parameters

The subject matter of this section primarily deals with an impact of the errors in the multilayer coating parameters on its spectral properties. As a rule, the errors are small enough, so it is possible to estimate the influence of the changes in individual parameters by calculating the spectral coefficient derivatives with respect to the corresponding parameter. The first item of the section asserts there are sufficiently simple algorithms allowing to calculate spectral coefficient derivatives very precisely and of low calculation efforts. The significance of this result goes beyond the limits of this section. The possibility of an absolutely precise and fast calculation of spectral coefficients derivatives is of vital importance for design methods of synthesis of multilayer coatings.

The major problem in estimating the impact of errors on multilayer coating properties lies in the fact that the concrete values of parameter variations are unknown. Usually, we may suggest only some approximate mean values of errors. A question rises in this connection of how to estimate possible spectral coefficient variations. One of the ways is a multiple reiterated modelling of concrete values of the parameter errors and a subsequent calculation of corresponding spectral coefficients. However, the number of different options in modelling the errors is so great, that significant difficulties appear in interpreting the obtained results. In this section we consider another approach to the estimation of the error impact (Grishina and Tikhonravov, 1985).

In a number of cases reliable quality conclusions can be made concerning the error influence on the spectral properties of coatings of various types. Here we discuss some general results to this effect.

### 1.4.1 Spectral coefficients derivatives with respect to layer parameters

Let $m$ be the number of layers. In a common case, such a multilayer coating is described by $3 m$-parameters: layer thicknesses $d_{j}$, refractive indices $n_{j}$, and the extinction coefficients $\kappa_{j}$. Note, that it is more convenient sometimes to consider complex refractive indices $\tilde{n}_{j}=n_{j}-i \kappa_{j}$, than refractive indices and extinction coefficients separately.

Let us designate one of the coating parameters as $x$ and a spectral coefficient as $\varphi$. In order to find a spectral coefficient derivative with respect to $x$, we can calculate the spectral coefficient with the changed parameter value of $x+\Delta x$ and make use of approximate formula

$$
\begin{equation*}
\frac{\partial \varphi}{\partial x}=\frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x} . \tag{1.4.1}
\end{equation*}
$$

In order to find all derivatives in this way, it is necessary to make $3 m+1$ calculations of the spectral coefficient, once for the initial values of the parameters and then $3 m$ times altering parameters one by one.

Generally speaking, the smaller $\Delta x$, the higher the accuracy of formula (1.4.1). However, any computer makes calculations with some final accuracy level. Due to this, the formula error can be significant: with $\Delta x$ decreasing, a relative calculation error in the numerator in (1.4.1) grows. The derivative calculation error proves to be most essential, if a relative change of the spectral coefficient at parameter variation is significant.

Below we describe another way of calculating spectral coefficient derivatives permitting to carry out calculations with computer accuracy. Besides it is more economical from the point of view of computation time. Let us now turn to the matrix method (see Section 1.2). In order to determine the amplitude transmittance and reflectance it is necessary to:

1. Form the characteristic matrices of the layers

$$
M_{p}=\left(\begin{array}{cc}
\cos \varphi_{j} & \frac{i}{q_{j}} \sin \varphi_{j}  \tag{1.4.2}\\
i q_{j} \sin \varphi_{j} & \cos \varphi_{j}
\end{array}\right) .
$$

where $\varphi_{j}=k \tilde{n}_{j} \cos \gamma_{j} d_{j}, q_{j}=\tilde{n}_{j} \cos \gamma_{j}$ in the $S$-case and $q_{j}=\tilde{n}_{j} / \cos \gamma_{j}$ in the $P$-case, and the $\gamma_{j}$ angles are determined in compliance with the Snell law: $n_{j} \sin \gamma_{j}=\alpha=n_{a} \sin \gamma_{a}$.
2. Find the characteristic matrix for the whole of the coating

$$
\begin{equation*}
M=M_{m} M_{m-1} \ldots M_{1} . \tag{1.4.3}
\end{equation*}
$$

3. Calculate the amplitude coefficients through the $M$-matrix elements according to the formulas

$$
\begin{equation*}
t=\frac{2 q_{a}}{q_{a} m_{11}+q_{s} m_{22}+q_{s} q_{a} m_{12}+m_{21}}, \tag{1.4.4}
\end{equation*}
$$

$$
\begin{equation*}
r=\frac{q_{a} m_{11}-q_{s} m_{22}+q_{s} q_{a} m_{12}-m_{21}}{q_{a} m_{11}+q_{s} m_{22}+q_{s} q_{a} m_{12}+m_{21}} . \tag{1.4.5}
\end{equation*}
$$

Let us first find spectral coefficient derivatives with respect to the layer thicknesses. The amplitude coefficients are composite functions of the layer parameters depending on them through the characteristic matrix elements. So, the amplitude coefficient derivatives are expressed through corresponding derivatives of the matrix elements. Differentiating (1.4.4). (1.4.5) we obtain

$$
\begin{align*}
& \frac{\partial t}{\partial d_{j}}=-\frac{t^{2}}{2 q_{a}}\left[q_{a} \frac{\partial m_{11}}{\partial d_{j}}+q_{s} \frac{\partial m_{22}}{\partial d_{j}}+q_{s} q_{a} \frac{\partial m_{12}}{\partial d_{j}}+\frac{\partial m_{21}}{\partial d_{j}}\right]  \tag{1.4.6}\\
& \frac{\partial r}{\partial d_{j}}=\frac{t}{2 q_{a}}\left[q_{a}(1-r) \frac{\partial m_{11}}{\partial d_{j}}-q_{s}(1+r) \frac{\partial m_{22}}{\partial d_{j}}+q_{s} q_{a}(1-r) \frac{\partial m_{12}}{\partial d_{j}}-(1+r) \frac{\partial m_{21}}{\partial d_{j}}\right] \tag{1.4.7}
\end{align*}
$$

According to (1.4.3), the derivatives of the characteristic matrix elements are determined from the equation:

$$
\begin{equation*}
\frac{\partial M}{\partial d_{j}}=M_{m} \ldots M_{j+1} \frac{\partial M_{j}}{\partial d_{j}} M_{j-1} \ldots M_{1} \tag{1.4.8}
\end{equation*}
$$

Let us designate the matrix $\partial M_{j} / \partial d_{j}$ as $D_{j}$. Differentiating (1.3.2), we find

$$
D_{j}=\frac{\partial M}{\partial d_{j}}=k \tilde{n}_{j} \cos \gamma_{j}\left(\begin{array}{cc}
-\sin \varphi_{j} & \frac{i}{q_{j}} \cos \varphi_{j}  \tag{1.4.9}\\
i q_{j} \cos \varphi_{j} & -\sin \varphi_{j}
\end{array}\right)
$$

Let us introduce two matrices

$$
\begin{align*}
& \Psi_{t}=-\frac{t^{2}}{2 q_{a}}\left(\begin{array}{cc}
q_{a} & 1 \\
q_{s} q_{a} & q_{s}
\end{array}\right), \\
& \Psi_{r}=\frac{t}{2 q_{a}}\left(\begin{array}{cc}
q_{a}(1-r) & -(1+r) \\
q_{s} q_{a}(1-r) & -q_{s}(1+r)
\end{array}\right) . \tag{1.4.10}
\end{align*}
$$

With their help, expressions (1.4.6), (1.4.7) can be written down as

$$
\begin{align*}
& \frac{\partial t}{\partial d_{j}}=T_{r}\left(\frac{\partial M}{\partial d_{j}} \Psi_{t}\right), \\
& \frac{\partial r}{\partial d_{j}}=T_{r}\left(\frac{\partial M}{\partial d_{j}} \Psi_{r}\right) \tag{1.4.11}
\end{align*}
$$

The symbol $T_{r}(\cdot)$ denotes here the trace of the matrix, which, according to the definition, is the sum of its diagonal elements. Multiplying the matrix

$$
\frac{\partial M}{\partial d_{j}}=\left(\begin{array}{ll}
\frac{\partial m_{11}}{\partial d_{j}} & \frac{\partial m_{12}}{\partial d_{j}} \\
\frac{\partial m_{21}}{\partial d_{j}} & \frac{\partial m_{22}}{\partial d_{j}}
\end{array}\right)
$$

by matrices (1.4.10), we prove the validity of the record (1.4.11). Substituting then (1.4.8) into (1.4.11) we obtain

$$
\begin{align*}
& \frac{\partial t}{\partial d_{j}}=T_{r}\left(M^{j+1} D_{j} \Psi_{t}^{j-1}\right),  \tag{1.4.12}\\
& \frac{\partial r}{\partial d_{j}}=T_{r}\left(M^{j+1} D_{j} \Psi_{r}^{j-1}\right) .
\end{align*}
$$

where $M^{j+1}, \Psi_{t}^{j-1}, \Psi_{r}^{j-1}$ denote the following matrix products:

$$
\begin{align*}
& M^{j+1}=M_{m} \ldots M_{j+1},  \tag{1.4.13}\\
& \Psi_{t}^{j-1}=M_{j-1} \ldots M_{1} \Psi_{t},  \tag{1.4.14}\\
& \Psi_{r}^{j-1}=M_{j-1} \ldots M_{1} \Psi_{r} . \tag{1.4.15}
\end{align*}
$$

Expressions (1.4.12) for the amplitude coefficients derivatives are absolutely accurate. It is easy to carry out calculations through a series of a certain type matrix products employing those expressions. Let us note that the matrix products (1.4.13) are obtained in the course of the coating characteristic matrix calculations, the matrix being necessary to find the amplitude coefficients proper. All the matrix products (1.4.14), (1.4.15) are calculated at a one-time pass along the layers of the coating. Due to this, the derivative calculation algorithm is highly economical.

Calculating derivatives with respect to other parameters is implemented in a similar way. It is most convenient first to find the derivatives with respect to the complex refractive index. It evidently results in similar (1.4.12) expressions:

$$
\begin{align*}
& \frac{\partial t}{\partial \tilde{n}_{j}}=T_{r}\left(M^{j+1} N_{j} \Psi_{t}^{j-1}\right),  \tag{1.4.16}\\
& \frac{\partial r}{\partial \tilde{n}_{j}}=T_{r}\left(M^{j+1} N_{j} \Psi_{r}^{j-1}\right) .
\end{align*}
$$

were $N_{j}$ are derivatives of the layer characteristic matrices with respect to $\tilde{n}_{j}$. In calculating these derivatives, it should be taken into account that the $\gamma_{j}$ angles are implicitly dependent on $\tilde{n}_{j}$. So, it is better to preliminary represent $\varphi_{j}$ and $q_{j}$ as follows

$$
\begin{aligned}
& \varphi_{j}=k\left(\tilde{n}_{j}^{2}-\alpha^{2}\right)^{1 / 2} d_{j}, \\
& q_{j}=\left(\tilde{n}_{j}^{2}-\alpha^{2}\right)^{1 / 2} \quad(\text { for the } S \text {-case) }, \\
& q_{j}=\frac{\tilde{n}_{j}^{2}}{\left(\tilde{n}_{j}^{2}-\alpha^{2}\right)^{1 / 2}} \quad \text { (for the } P \text {-case). }
\end{aligned}
$$

After a number of non- complicated transformations $\partial M_{j} / \partial \tilde{n}_{j}$ derivatives take the form

$$
N_{j}=\frac{\partial M_{j}}{\partial \tilde{n}_{j}}=\frac{d_{j}}{\tilde{n}_{j} \cos ^{2} \gamma_{j}} D_{j}+i \frac{\cos \varphi_{j}}{\cos \gamma_{j}}\left(\begin{array}{cc}
0 & -\frac{1}{q_{j}^{2}} \\
1 & 0
\end{array}\right)
$$

for the $S$-case, while for the $P$-case they take the form:

$$
N_{j}=\frac{\partial M_{j}}{\partial \tilde{n}_{j}}=\frac{d_{j}}{\tilde{n}_{j} \cos ^{2} \gamma_{j}} D_{j}+i \frac{\cos \varphi_{j}}{\cos \gamma_{j}}\left(1-\tan ^{2} \gamma_{j}\right)\left(\begin{array}{cc}
0 & -\frac{1}{q_{j}^{2}} \\
1 & 0
\end{array}\right)
$$

where $D_{j}$ is a matrix determined by the expression (1.4.9).
As $\tilde{n}_{j}=n_{j}-i \kappa_{j}$, the amplitude coefficients derivatives with respect to the real parameters $n_{j}, \kappa_{j}$ are connected with the derivatives (1.4.16) through the simplest expressions:

$$
\begin{array}{lll}
\frac{\partial t}{\partial n_{j}}=\frac{\partial t}{\partial \tilde{n}_{j}}, & \frac{\partial r}{\partial n_{j}}=\frac{\partial r}{\partial \tilde{n}_{j}}  \tag{1.4.17}\\
\frac{\partial t}{\partial \kappa_{j}}=-i \frac{\partial t}{\partial \tilde{n}_{j}}, & \frac{\partial r}{\partial \kappa_{j}}=-i \frac{\partial r}{\partial \tilde{n}_{j}}
\end{array}
$$

It is very easy to express the transmittance and reflectance derivatives most frequently required in calculations, through amplitude coefficient derivatives. Since $R=|r|^{2}=r r^{*}$, then.

$$
\frac{\partial R}{\partial d_{j}}=\frac{\partial r}{\partial d_{j}} r^{*}+r \frac{\partial r^{*}}{\partial d_{j}}=2 \operatorname{Re}\left\{r^{*} \frac{\partial r}{\partial d_{j}}\right\} .
$$

Similarly,

$$
\frac{\partial T}{\partial d_{j}}=2 \frac{q_{s}}{q_{a}} \operatorname{Re}\left\{t^{*} \frac{\partial t}{\partial d_{j}}\right\}
$$

Other transmittance and reflectance derivatives are expressed in exactly the same way. Taking account of (1.4.17), we obtain

$$
\begin{array}{ll}
\frac{\partial T}{\partial n_{j}}=2 \frac{q_{s}}{q_{a}} \operatorname{Re}\left\{t^{*} \frac{\partial t}{\partial \tilde{n}_{j}}\right\}, & \frac{\partial R}{\partial n_{j}}=2 \operatorname{Re}\left\{r^{*} \frac{\partial r}{\partial \tilde{n}_{j}}\right\}, \\
\frac{\partial T}{\partial \kappa_{j}}=2 \frac{q_{s}}{q_{a}} \operatorname{Im}\left\{t^{*} \frac{\partial t}{\partial \tilde{n}_{j}}\right\}, & \frac{\partial R}{\partial \kappa_{j}}=2 \operatorname{Im}\left\{r^{*} \frac{\partial r}{\partial \tilde{n}_{j}}\right\} .
\end{array}
$$

Note in conclusion, that higher derivatives of multilayer coating spectral coefficients can be obtained following this pattern.

### 1.4.2 Impact of the layer parameter errors on the multilayer coatings spectral coefficients: probability assessment method

Let us note that the very values of the spectral coefficient derivatives give important information on the nature of error influence on the coating spectral properties. Errors of the same value can grow in significance wherever the derivatives are greater. Spectral coefficient derivatives depend on the wavelength (the wavenumber), and thus are themselves certain spectral characteristics of a coating: the influence of the errors will be stronger in the spectral band where these derivatives are greater.

At the same time, the set of derivatives cannot of itself give sufficient representation of the total influence of all possible errors, since, first, the changes caused by the errors interact in a very complicated way, and, second, the concrete values of the errors are frequently unknown in advance.

An aggregated influence of the errors can be taken account of by putting down the Taylor expansion of the spectral coefficient increments. For reasons of certainty, let us further consider the transmittance coefficient. Let us also hold all layers non-absorbing, i.e., consider dependence of $T$ on $2 m$ parameters - layer thicknesses and refractive indices. Let $T^{0}(k)$ be a transmittance corresponding to some given set of parameters $d_{1}, \ldots, d_{m}, n_{1}, \ldots, n_{m}$. Let us designate the thickness and refractive index variations of the $j$-th layer $\xi_{j}, \eta_{j}$. As a rule, these variations (errors) are small, and the related transmittance coefficient change approximates to a high degree of accuracy by a segment of the Taylor series:

$$
\begin{equation*}
\Delta T(k)=T(k)-T^{0}(k) \approx \sum_{j=1}^{m} \frac{\partial T}{\partial d_{j}} \xi_{j}+\sum_{j=1}^{m} \frac{\partial T}{\partial n_{j}} \eta_{j}+\frac{1}{2} \sum_{i, j=1}^{m}\left\{\frac{\partial^{2} T}{\partial d_{i} \partial d_{j}} \xi_{i} \xi_{j}+\frac{\partial^{2} T}{\partial n_{i} \partial n_{j}} \eta_{i} \eta_{j}+2 \frac{\partial^{2} T}{\partial d_{i} \partial n_{j}} \xi_{i} \eta_{j}\right\} . \tag{1.4.18}
\end{equation*}
$$

If the concrete values of the errors in the layer parameters were known, we could find the corresponding changes of the coefficients from the known coefficient derivatives using (1.4.18)type formulas. However, we can talk only about some statistical level of errors.

Errors in the layers parameters are caused by numerous different factors. Those are monitoring methods errors, deviations in the parameters of materials used for sputtering, degree of cleanliness of the chamber, the temperature of the substrate, etc. Errors caused by various factors are practically random. It is widely known that in similar situations a distribution of the summed up errors is quite accurately described by the normal distribution law. So, let us consider each $\xi_{j}, \eta_{j}$ as distributed in accordance with the normal law with zero mathematical expectation and some given standard deviation. Assume also that errors in the parameters of individual layers are noncorrelated. This assumption best corresponds to deposition using quartz monitoring method. Let us designate the standard deviations of errors in the layer thicknesses $\sigma_{d, j}$, standard deviations of errors in the refractive indices $\sigma_{n, j}$.

A change in the transmittance $\Delta T(k)$ depends on $\xi_{j}, \eta_{j}$ and, consequently, is a random value. The $\Delta T(k)$ distribution can be characterized by a mathematical expectation and a standard deviation. Both these values also depend on the wavenumber $k$. Let us designate them as $M_{\Delta T}(k)$ and $S_{\Delta T}(k)$, respectively. For random function of the (1.4.18)-type, general formulas can be deduced, expressing their mathematical expectation and the standard deviation through the $\xi_{j}, \eta_{j}$ standard deviations:

$$
\begin{align*}
& M_{\Delta T}(k)=\frac{1}{2} \sum_{j=1}^{m}\left(\frac{\partial^{2} T}{\partial d_{j}^{2}} \sigma_{d, j}^{2}+\frac{\partial^{2} T}{\partial n_{j}^{2}} \sigma_{n, j}^{2}\right)  \tag{1.4.19}\\
& S_{\Delta T}^{2}(k)=\sum_{j=1}^{m}\left(\left(\frac{\partial T}{\partial d_{j}}\right)^{2} \sigma_{d, j}^{2}+\left(\frac{\partial T}{\partial n_{j}}\right)^{2} \sigma_{n, j}^{2}\right)+  \tag{1.4.20}\\
& +\frac{1}{2} \sum_{i, j=1}^{m}\left\{\left(\frac{\partial^{2} T}{\partial d_{t} \partial d_{j}}\right)^{2} \sigma_{d, i}^{2} \sigma_{d, j}^{2}+\left(\frac{\partial^{2} T}{\partial n_{t} \partial n_{j}}\right)^{2} \sigma_{n, i}^{2} \sigma_{n, j}^{2}+\left(\frac{\partial^{2} T}{\partial d_{t} \partial n_{j}}\right)^{2} \sigma_{d, i}^{2} \sigma_{n, j}^{2}\right\}
\end{align*}
$$

The values for $\sigma_{d, j}, \sigma_{n, j}$ in (1.4.19), (1.4.20) characterize the error levels in the thicknesses and the refractive indices. The $M_{\Delta T}(k)$ and $S_{\Delta T}(k)$ functions calculated on their basis allow to vividly characterize possible deviations of transmittance from the calculated $T^{0}(k)$ curve. This characteristic of the most likely values of $T(k)$ is provided by the curve of
transmittance mathematical expectation $T^{0}(k)+M_{\Delta T}(k)$ with the standard deviation corridor $T^{0}(k)+M_{\Delta T}(k) \pm S_{\Delta T}(k)$ marked on the figure.


Figure 1.10: Corridor of the most likely trancmittance values of four-layer antireflection coating: calculated transmittance $T^{0}(k)$ (solid curve), mathematical expectation $T^{0}(k)+M_{\Delta T}(k)$ (dashed curve)

Figure 1.10 illustrates the error sensibility of four-layer antireflection coating at normal light incidence. The calculated parameters have the following values: odd layers refractive indices are equal to 2.00 , while even layers to 1.45 , the layer thicknesses are $16.2,33.6,141.8$, 93.3 nm . The substrate refractive index is equal to 1.52 . The following values were assumed for possible error standard deviations in the parameters of the layers: $\sigma_{d, 1}=\sigma_{d, 2}=2 \mathrm{~nm}, \sigma_{d, 3}=\sigma_{d, 4}=5 \mathrm{~nm}, \sigma_{n}=0.02$ for all layers. Thus, errors in layer thicknesses are about $4-6 \%$ of the thicknesses of the second through fourth layers and $12 \%$ of the first layer thickness. Figure 1.10 provides a calculated transmittance of the antireflection coating $T^{0}(\mathrm{k})$ (solid curve) and its mathematical expectation $T^{0}(k)+M_{\Delta T}(k)$ (dashed curve) with the marked corridor of the standard deviation. This corridor characterizes a possible deviation of the transmittance from the calculated curve.

Note, that expressions (1.4.19), (1.4.20) were obtained in the assumption of noncorrelated errors in parameters of different layers. This assumption is quite justified in coating production
with quartz resonator monitoring. It is well known that in employing the optical monitoring methods error autocompensation often occurs in layer thicknesses, that is the latter are not independent random values. The probability approach described above can be expanded to include such cases by introducing a correlation matrix for random variations of layer thicknesses.

### 1.4.3 Multilayer coating spectral properties: influence of separate layer parameter variations

In this subsection we will obtain some qualitative conclusions concerning the influence of separate parameter variations on multilayer coating properties. A number of formulas describing the energy coefficients changes with a change in a thickness or refractive index of a layer serve as the basis for these conclusions. To simplify the considerations, we will not supply a detailed deduction of the formulas but give its brief explanation instead.

Let us single out some layer of a non-absorbing multilayer coating and designate its thickness as $d$ and its refractive index as $n$. The set of layers preceding the layer in question will be termed subsystem 1, while the set of layers following it will be referred to as subsystem 2 (see Fig. 1.9 in the previous section). Further we will use the same designations as in Section 1.3: spectral coefficients of subsystems 1 and 2 are designated with proper indices, spectral coefficients and phase changes on the reflection for wave propagating in the reverse direction (i.e., from the substrate to the outer space) are marked with a prime.

The initial expressions for the formulas deduced below are those (1.3.31) and (1.3.32) from the previous section. With their help amplitude coefficient increments (and then energy coefficient increments) can be expressed through the inner layer parameter change. It ought to be taken into account that with a change of the refractive index or introduction of a small absorption in the layer not only phase thickness $\varphi$ changes, but the subsystem amplitude coefficients change too, due to the change of parameters of the framing media. Going through rather a tedious procedure of calculations, we will find as a result that with a change of the refractive index of the singled out layer by $\Delta n$, reflectance and transmittance will get increments
$\Delta R=-\Delta T=2 \Delta n \frac{T^{2} \sqrt{R_{1} R_{2}}}{n T_{1} T_{2}}\left\{2 \varphi \sin 2 \theta+\left[\frac{1+R_{1}}{\sqrt{R_{1}}} \sin \left(\varphi-\Delta_{2}^{\prime}\right)+\frac{1+R_{2}}{\sqrt{R_{2}}} \sin \left(\varphi-\Delta_{1}\right)\right] \sin \varphi\right\}$.
Here, like in Section 1.3, $\theta=\frac{1}{2}\left(\Delta_{1}+\Delta_{2}^{\prime}-2 \varphi\right)$.

Energy coefficient changes caused by taking into account small absorption in the particular layer under consideration are equal to

$$
\begin{align*}
& \Delta T=-2 \kappa \frac{T^{2}}{n}\left\{\frac{1-R_{1} R_{2}}{T_{1} T_{2}} \varphi+\left[\frac{\sqrt{R_{1}}}{T_{1}} \sin \left(\varphi-\Delta_{1}\right)+\frac{\sqrt{R_{2}}}{T_{2}} \sin \left(\varphi-\Delta_{2}^{\prime}\right)\right] \sin \varphi\right\} .  \tag{1.4.22}\\
& \Delta R=2 \kappa \frac{T}{n}\left\{\frac{R_{2}-R_{1}-R\left(1-R_{1} R_{2}\right)}{T_{1} T_{2}} \varphi+\left[(1-R) \frac{\sqrt{R_{2}}}{T_{2}} \cos \left(\varphi-\Delta_{2}^{\prime}\right)-(1-R) \frac{\sqrt{R_{1}}}{T_{1}} \cos \left(\varphi-\Delta_{1}\right)\right] \sin \varphi\right\} . \tag{1.4.23}
\end{align*}
$$

Parameter $\kappa$ in (1.4.22), (1.4.23) is an imaginary part of the refractive index (extinction coefficient) of the given layer. Taking account of small absorption in the layer, absorptance $A$ differs from 0 . Since, in our assumption, the other layers of the coating are non-absorbing, $A=-(\Delta T+\Delta R)$ and we obtain from (1.4.22), (1.4.23) the following:

$$
\begin{equation*}
A=2 \kappa \frac{T}{n T_{1}}\left\{\left(1+R_{1}\right) \varphi+2 \sqrt{R_{1}} \cos \left(\varphi-\Delta_{1}\right) \sin \varphi\right\} . \tag{1.4.24}
\end{equation*}
$$

Lastly, with the change of the layer thickness by a small value, energy coefficients receive increments

$$
\begin{equation*}
\Delta T=-\Delta R=4 \Delta d \frac{T^{2} \sqrt{R_{1} R_{2}}}{T_{1} T_{2}} k n \cos \gamma \sin 2 \theta \tag{1.4.25}
\end{equation*}
$$

Let us consider the conclusions following from (1.4.21) through (1.4.25). Factor $T^{2}$ is a part of the expression (1.4.21), where $T$ is transmittance for the whole system. So, variations of the refractive indices would tell but insignificantly on the spectral properties of a multilayer coating in the areas of high reflectance. This conclusion is further proved by experimental data: it is known that refractive indices deviations mostly tell on the position of boundaries for a high reflectance region, but practically do not influence on the $R$ value inside the region.

Similarly, it follows from (1.4.22), (1.4.23) that small absorption in layers hardly changes the transmittance in high reflection regions, but results in a significantly more noticeable change of the reflectance since $\Delta T / \Delta R \sim 1 / T$. As shown in (1.4.24), the same extinction coefficient of the layer causes a greater increase of $A$ with less transmittance of the subsystem following the absorbing layer. Thus, in mirrors, the greatest contribution into the total absorption will be made by outer layers. Note, lastly, that on certain conditions separate layers of a multilayer coating may happen to be highly absorbing even at small $\kappa$. It is connected with the fact that the ratio $T / T_{1}$, being a part of (1.4.24) may be great. Really, according to (1.3.34)

$$
\begin{equation*}
\frac{T}{T_{1}}=\frac{T_{2}}{1+R_{1} R_{2}-2 \sqrt{R_{1} R_{2}} \cos ^{2} \theta}=\frac{T_{2}}{\left(1-\sqrt{R_{1} R_{2}}\right)^{2}+4 \sqrt{R_{1} R_{2}} \sin ^{2} \theta} . \tag{1.4.26}
\end{equation*}
$$

As seen from (1.4.26), if $\theta$ for any layer is divisible by $\pi$, and the reflectances from the framing subsystems are close to unit, the $T / T_{1}$ ratio is large. Such layers serve as kind of resonators: the specified conditions for $R_{1}, R_{2}, \theta$ provide a resonant accumulation of the wave energy in the layer due to multiple reflections from its boundaries and phase condition

$$
\Delta_{1}+\Delta_{2}^{\prime}-2 \varphi=0(\bmod 2 \pi)
$$

It is exactly the situation that exists for the central layer of a narrow bandpass interference filter. So, even a small absorption of the material of the central layer brings about a significant deterioration of the filter properties.

Let us turn now to formula (1.4.25). Note that if $\theta$ is divisible by $\pi$, the layer thickness variations in a first approximation have no influence at all on the values of the transmittance and reflectance. This condition is satisfied at the central wavelength for all layers of quarter-wave dielectric mirrors and narrow bandpass filters. So, the deviations of spectral characteristics from the calculated characteristics can be primarily accounted for by the refractive indices variations and the absorption in the layers.

Expression (1.4.25) supports a conclusion that layer thicknesses variations tell on the spectral characteristics significantly less in the high reflection region than in other regions. This fact is revealed in refining methods of multilayer coating synthesis. It turns out that multilayer mirrors used as starting designs in synthesis may significantly change the spectral characteristics beyond the reflection region due to layer thicknesses variations causing almost no changes of the $R$-value within the region. In this way edge filters, for instance, are synthesized.

### 1.5 Multilayer periodic systems. Quarter-wave dielectric mirrors

Multilayer periodic systems are of special interest primarily because they include quarterwave dielectric mirrors and multilayer interference filters contain periodic layer subsystems. The periodicity property allows to express the system characteristic matrix through Chebyshev polynomials.

Due to a wide application of quarter-wave mirrors and interference filters, an in-depth study of their spectral properties, an analysis of the layer parameters errors impact, a research of
the influence of an absorption in dielectric layers are essential. Due to special structures of the above systems, approximate analytical expressions describing the dependence of their major spectral properties on the layer parameters are possible to obtain. These expressions are of great practical value since they provide an easy qualitative and quantitative analysis of mirror and filter properties.

### 1.5.1 Characteristic matrix of a periodic multilayer system

Let us consider a two-material periodic multilayer system consisting of repetitive layer pairs with $n_{1}$ and $n_{2}$ refractive indices and $d_{1}$ and $d_{2}$ thicknesses. Let us designate the number of periods as $N$. Thus the total number of the layers is equal to $2 N$. Layers can also be absorbing. In this case their complex refractive indices are denoted as $\tilde{n}_{1}$ and $\tilde{n}_{2}$. Let us designate the product of characteristic matrices making up a period as $A$. In the general case, taking account of (1.2.9), we have

$$
\begin{gather*}
A=M_{2} M_{1}=\left(\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right),  \tag{1.5.1}\\
a_{11}=\cos \varphi_{1} \cos \varphi_{2}-\frac{q_{1}}{q_{2}} \sin \varphi_{1} \sin \varphi_{2}, \\
a_{12}=\frac{i}{q_{1}} \sin \varphi_{1} \cos \varphi_{2}+\frac{i}{q_{2}} \cos \varphi_{1} \sin \varphi_{2}, \\
a_{21}=i q_{1} \sin \varphi_{1} \cos \varphi_{2}+i q_{2} \cos \varphi_{1} \sin \varphi_{2}, \\
a_{22}=\cos \varphi_{1} \cos \varphi_{2}-\frac{q_{2}}{q_{1}} \sin \varphi_{1} \sin \varphi_{2},
\end{gather*}
$$

where $\varphi_{1}=k \tilde{n}_{1} \cos \gamma_{1} d_{1}, \varphi_{2}=k \tilde{n}_{2} \cos \gamma_{2} d_{2}$,

$$
\begin{aligned}
& q_{1}=\tilde{n}_{1} \cos \gamma_{1}, \quad q_{2}=\tilde{n}_{2} \cos \gamma_{2}, \quad(\text { in the } S \text {-case }), \\
& q_{1}=\frac{\tilde{n}_{1}}{\cos \gamma_{1}}, \quad q_{2}=\frac{\tilde{n}_{2}}{\cos \gamma_{2}}, \quad(\text { in the } P \text {-case). }
\end{aligned}
$$

The characteristic matrix of the whole system equals the $N$-th power of the $A$ matrix:

$$
\begin{equation*}
M=\left(M_{2} M_{1}\right)^{N}=A^{N} . \tag{1.5.2}
\end{equation*}
$$

The Chebyshev polynomials allow to express the characteristic matrix elements with the help of the $A$-period matrix elements. Before we show it, let us define the Chebyshev polynomials and consider some of their properties. Strictly speaking, the polynomials we are
interested in are known as second-type Chebyshev polynomials. Since we are not going to tackle polynomials of any other type, we will further use only the short-version name for them everywhere, except in the definition.

It is easiest to define the $U_{n}(a)$ Chebyshev polynomials of the second type with the help of the recurrent formula expressing an $n$-th order polynomial through the $n-1$ and $n-2$ order polynomials:

$$
\begin{equation*}
U_{n}(a)=2 a U_{n-1}(a)-U_{n-2}(a) . \tag{1.5.3}
\end{equation*}
$$

The two first polynomials are set as the initial values for the recurrent formula (1.5.3):

$$
U_{0}(a)=1, \quad U_{1}(a)=2 a .
$$

With the help of the formula (1.5.3) it is possible to find consequently a Chebyshev polynomial of any power. Let us write out few first ones:

$$
\begin{aligned}
& U_{2}(a)=4 a^{2}-1, \\
& U_{3}(a)=8 a^{3}-4 a, \\
& U_{4}(a)=16 a^{4}-12 a^{2}+1, \\
& U_{5}(a)=32 a^{5}-32 a^{3}+6 a .
\end{aligned}
$$

Note, that all polynomials of the even power are even functions, while those of the odd power are odd functions.

Let us emphasize a useful property of the second type Chebyshev polynomials. Let us introduce the function

$$
\begin{equation*}
F(t, a)=\frac{1}{1-2 t a+t^{2}} \tag{1.5.4}
\end{equation*}
$$

It turns out that, expanded into the Taylor series with respect to $t$, the function will reveal coefficients which are $U_{n}(a)$ polynomials:

$$
F(t, a)=\sum_{n=0}^{\infty} U_{n}(a) t^{n} .
$$

Due to this property, function (1.5.4) is known as a second type Chebyshev polynomial generating function.

With the help of Chebyshev polynomials, the $M$-matrix, being the $N$-th power of the $A$-matrix, is expressed through the elements of the latter according to the formula

$$
M=A^{N}=\left(\begin{array}{cc}
a_{11} U_{N-1}(a)-U_{N-2}(a) & a_{12} U_{N-1}(a)  \tag{1.5.5}\\
a_{21} U_{N-1}(a) & a_{22} U_{N-1}(a)-U_{N-2}(a)
\end{array}\right),
$$

where the $a$-argument of polynomials is a half-sum of the diagonal elements of the $A$-matrix:

$$
\begin{equation*}
a=\frac{a_{11}+a_{22}}{2} . \tag{1.5.6}
\end{equation*}
$$

The simplest way to verify the result is to use the mathematical induction method. It is valid at $N=1$, if $U_{-1}(a)=0$, which does not contradict to the recurrent formula (1.5.3). Assume that formula (1.5.5) is valid for the $N$-th power of the $A$-matrix, and let us prove that a similar formula is also valid for the $N+1$-st power. Multiplying matrix (1.5.5) by A we obtain:

$$
M=A^{N+1}=\left(\begin{array}{cc}
g_{1} U_{N-1}-a_{11} U_{N-2} & a_{12}\left(2 a U_{N-1}-U_{N-2}\right)  \tag{1.5.7}\\
a_{21}\left(2 a U_{N-1}-U_{N-2}\right) & g_{2} U_{N-1}-a_{22} U_{N-2}
\end{array}\right),
$$

where $g_{1}=a_{11}^{2}+a_{12} a_{21}, \quad g_{2}=a_{12} a_{21}+a_{22}^{2}$.
To reduce the record we omitted here the Chebyshev polynomial arguments and, besides, took account of equation (1.5.6).

All the determinants of the layers characteristic matrices equal a unit. So is the determinant of any product of these matrices, the matrix A determinant, in particular. Thus,

$$
a_{11} a_{22}-a_{12} a_{21}=1
$$

Expressing $a_{12}, a_{21}$ from here, we transform the diagonal elements of matrix (1.5.7) into

$$
\begin{aligned}
& \left(a_{11}^{2}+a_{11} a_{22}-1\right) U_{N-1}-a_{11} U_{N-2}=a_{11}\left(2 a U_{N-1}-U_{N-2}\right)-U_{N-1} \\
& \left(a_{11} a_{22}-1+a_{22}^{2}\right) U_{N-1}-a_{22} U_{N-2}=a_{22}\left(2 a U_{N-1}-U_{N-2}\right)-U_{N-1} .
\end{aligned}
$$

Now, with the recurrent formula (1.5.3) taken into account, the matrix (1.5.7) is written down in a way similar to (1.5.5):

$$
A^{N+1}=\left(\begin{array}{cc}
a_{11} U_{N}(a)-U_{N-1}(a) & a_{12} U_{N-1}(a) \\
a_{21} U_{N-1}(a) & a_{22} U_{N}(a)-U_{N-1}(a)
\end{array}\right) .
$$

Thus, the validity of the formula (1.5.5) is proved by induction.
Let us now turn our attention to the fact that in proving formula (1.5.5) we only made use of the A-matrix determinant's equality to a unit. The specific content of A was not emphasized as important. Thus, expression (1.5.5) is valid for any periodic system with an arbitrary number of layers in a period.

### 1.5.2 Quarter-wave mirror properties at the central wavelength

The $A$-matrix elements for a two-material periodic system are written out in (1.5.1) and the Chebyshev polynomial argument is equal to

$$
\alpha=\cos \varphi_{1} \cos \varphi_{2}-\frac{1}{2}\left(\frac{q_{1}}{q_{2}}+\frac{q_{2}}{q_{1}}\right) \sin \varphi_{1} \sin \varphi_{2} .
$$

Let us find with the help of (1.5.5) amplitude transmittance and reflectance of quarterwave dielectric mirror at the central wavelength $\lambda_{0}$. Let absorption be lacking and incidence normal. Here in (1.5.1) we have $\varphi_{1}=k n_{1} d_{1}, \varphi_{2}=k n_{2} d_{2}, q_{1}=n_{1}, q_{2}=n_{2}$. By the definition of the quarter-wave mirror

$$
n_{1} d_{1}=n_{2} d_{2}=\frac{\lambda_{0}}{4} .
$$

At the central wavelength $k=2 \pi / \lambda_{0}$ and $\varphi_{1}=\varphi_{2}=\pi / 2$. The $A$ matrix takes the form

$$
A=\left(\begin{array}{cc}
-\frac{n_{1}}{n_{2}} & 0 \\
0 & -\frac{n_{2}}{n_{1}}
\end{array}\right)
$$

and the argument (1.5.8) turns to be equal to

$$
a=-\frac{1}{2}\left(\frac{n_{1}}{n_{2}}+\frac{n_{2}}{n_{1}}\right) .
$$

Let us designate the ratio of refractive indices of even and odd layers as $b$ :

$$
b=\frac{n_{2}}{n_{1}} .
$$

Taking this into account, formula (1.5.5) gives

$$
M=\left(\begin{array}{cc}
-b^{-1} U_{N-1}-U_{N-2} & 0  \tag{1.5.8}\\
0 & -b U_{N-1}-U_{N-2}
\end{array}\right) .
$$

Due to a specific form of Chebyshev polynomials argument $a=-\left(b+b^{-1}\right) / 2$, the recording of the matrix elements can be significantly simplified. With this in view, let us consider the generating function (1.5.4). With $a=-\left(b+b^{-1}\right) / 2$ it can be represented as two simple co-factors:

$$
\frac{1}{1+t\left(b+b^{-1}\right)+t^{2}}=\frac{1}{1+b t} \frac{1}{1+b^{-1} t},
$$

each of them easily expanding into a series:

$$
\begin{aligned}
& \frac{1}{1+b t}=\sum_{m=0}^{\infty}(-1)^{m} b^{m} t^{m} \\
& \frac{1}{1+b^{-1} t}=\sum_{k=0}^{\infty}(-1)^{k} \frac{t^{k}}{b^{k}}
\end{aligned}
$$

By multiplying the sequences we obtain the expansion of the generating function (1.5.4):

$$
F\left[t,-\left(b+b^{-1}\right) / 2\right]=\sum_{n=0}^{\infty}\left[(-1)^{n} \sum_{m=0}^{\infty} b^{2 m-n}\right] t^{n} .
$$

Due to the generating function property, the coefficients are Chebyshev polynomials. It follows from here that

$$
U_{n}\left[-\left(b+b^{-1}\right) / 2\right]=(-1)^{n} \sum_{m=0}^{\infty} b^{2 m-n}=(-1)^{n}\left[b^{-n}+b^{-n+2}+\ldots+b^{n}\right]
$$

Using the equation we obtain that in (1.5.8)

$$
\begin{aligned}
& -b^{-1} U_{N-1}-U_{N-2}=(-1)^{N} b^{-N}, \\
& -b U_{N-1}-U_{N-2}=(-1)^{N} b^{N} .
\end{aligned}
$$

Thus the matrix (1.5.8) has a very simple form:

$$
M=(-1)^{N}\left(\begin{array}{cc}
b^{-N} & 0  \tag{1.5.9}\\
0 & b^{N}
\end{array}\right)
$$

This expression is obtained for a $2 N$-layer mirror. In order to obtain a characteristic matrix of a quarter-wave mirror with an odd number of layers, it is necessary to finally multiply the matrix (1.5.9) on the left by the matrix of the last layer.

This layer features the refractive index $n_{1}$, and its matrix at the central wavelength is equal to

$$
\left(\begin{array}{cc}
0 & -\frac{i}{n_{1}} \\
-i n_{1} & 0
\end{array}\right) .
$$

As a result we obtain the following expression for the characteristic matrix of the $2 N+1$-layer mirror:

$$
M=(-1)^{N+1}\left(\begin{array}{cc}
0 & \frac{i b^{N}}{n_{1}}  \tag{1.5.10}\\
i b^{-N} n_{1} & 0
\end{array}\right)
$$

Amplitude transmittance and reflectance are obtained from (1.2.13), (1.2.14). We have for a $2 N$-layer mirror:

$$
\begin{equation*}
t=\frac{(-1)^{N} 2 n_{a}}{n_{a} b^{-N}+n_{s} b^{N}}, \quad r=\frac{n_{a} b^{-N}-n_{s} b^{N}}{n_{a} b^{-N}+n_{s} b^{N}}, \tag{1.5.11}
\end{equation*}
$$

and for a $2 N+1$-layer mirror:

$$
\begin{equation*}
t=\frac{i(-1)^{N} 2 n_{a} n_{1}}{n_{s} n_{a} b^{N}+n_{1}^{2} b^{-N}}, \quad r=\frac{n_{s} n_{a} b^{N}-n_{1}^{2} b^{-N}}{n_{s} n_{a} b^{N}+n_{1}^{2} b^{-N}} . \tag{1.5.12}
\end{equation*}
$$

Remember, that in (1.5.11), (1.5.12) $b=n_{2} / n_{1}$ is a ratio of an even layers refractive index and odd layers refractive index.

Let us find what a phase change on reflection at the central wavelength is equal to. Let $b>1$, i.e., $n_{2}=n_{H}, n_{1}=n_{L}$. Note that a mirror with an even number of layers will have the outer layer refractive index $n_{H}$, while that with an odd number of layers $n_{L}$. With $b>1$ and a sufficiently large $N$, the numerator in the expression (1.5.11) for $r$ is negative, and in the expression (1.5.12) it is positive. So the phase change for the $2 N$-layer mirror is equal to $\pi$, and that for a $2 N+1$-layer mirror equals zero. With $b<1$ the signs in the numerators for $r$ will change, but simultaneously, the refractive indices of the outer layers will change, too: the low ones will grow high, and vice versa. It means that quarter-wave mirrors at the central wavelength irrespective of the number of layers, will have the phase change equal to $n$ with the outer layer refractive index equal to $n_{H}$, and the phase change equal to zero with the outer layer refractive index equal to $n_{L}$.

Let us determine now quarter-wave mirror reflectance at the central wavelength. Assume that first the number of layers is even and $b>1$. From (1.5.11) we obtain

$$
\begin{equation*}
R=\left[\frac{\left(n_{a} / n_{s}\right) b^{-2 N}-1}{\left(n_{a} / n_{s}\right) b^{-2 N}+1}\right]^{2} . \tag{1.5.13}
\end{equation*}
$$

It is worthwhile to represent the expression in the following approximate form:

$$
\begin{equation*}
R \approx 1-4 \frac{n_{a}}{n_{s}} b^{-2 N}=1-4 \frac{n_{a}}{n_{s}}\left(\frac{n_{L}}{n_{H}}\right)^{2 N} . \tag{1.5.14}
\end{equation*}
$$

Passing from (1.5.13) to (1.5.14) we discarded all the terms containing $b^{-4 N}$ factor. It is easy to find the proof of it by subtracting (1.5.14) from (1.5.13). For the overwhelming majority of mirrors practically employed, the $b^{-4 N}$ value is negligibly small. For instance, for a 10-layer mirror with refractive indices $n_{L}=1.35$ for odd and $n_{H}=2.30$ for even layers $b^{-4 N}=b^{-20} \approx 2 \cdot 10^{-5}$. So, the approximate expression (1.5.14) gives a high accuracy value of reflectance.

Similarly, with $b<1$, that is with $n_{1}=n_{H}, n_{2}=n_{L}$, by discarding terms of the $b^{-4 N}$ order, we obtain an approximate value for a $2 N$-layer mirror:

$$
\begin{equation*}
R \approx 1-4 \frac{n_{s}}{n_{a}} b^{2 N}=1-4 \frac{n_{s}}{n_{a}}\left(\frac{n_{L}}{n_{H}}\right)^{2 N} . \tag{1.5.15}
\end{equation*}
$$

With the same number of layers and the same high and low refractive indices expression (1.5.14) gives a greater value for $R$, as the outer space is usually the air and $n_{a}=1<n_{s}$. Thus a mirror with a high refractive index of the outer layer has a higher reflectance.

At an odd number of layers $2 N+1$, through similar simplification procedures, we obtain in the case of $b>1\left(n_{1}=n_{L}, n_{2}=n_{H}\right)$

$$
\begin{equation*}
R \approx 1-4 \frac{n_{1}^{2}}{n_{s} n_{a}} b^{-2 N}=1-4 \frac{n_{L}^{2}}{n_{s} n_{a}}\left(\frac{n_{L}}{n_{H}}\right)^{2 N} \tag{1.5.16}
\end{equation*}
$$

and in the case of $b<1\left(n_{1}=n_{H}, n_{2}=n_{L}\right)$

$$
\begin{equation*}
R \approx 1-4 \frac{n_{s} n_{a}}{n_{1}^{2}} b^{2 N}=1-4 \frac{n_{s} n_{a}}{n_{H}^{2}}\left(\frac{n_{L}}{n_{H}}\right)^{2 N} . \tag{1.5.17}
\end{equation*}
$$

In the second case, the refractive index of the odd layers is high and comparing (1.5.16) and (1.5.17) we can easily see that a high refractive index of the outer layer again reveals higher reflectance.

### 1.5.3 Width of the quarter-wave mirror high-reflectance zones

Let us investigate the width of the quarter-wave mirror high-reflectance zones at the normal incidence and small deviations of the incidence angle from zero. For this purpose, let us consider the matrix product of the period in the general form for the oblique incidence case. Let the layers of the mirror at this subsection be non-absorbing, and, correspondingly, the layers refractive indices real. The parameters of the layers making up a period are connected through the equation

$$
n_{1} d_{1}=n_{2} d_{2}=\frac{\lambda_{0}}{4} .
$$

where $\lambda_{0}$ is a certain fixed wavelength (central wavelength, in accordance with acknowledged terminology).

First of all, it is necessary to determine what mirror high-reflectance zones are and what their boundaries are. As is known, as the number of periods grows, the residual transmittance in the high-reflectance zones decreases, and the $R(\lambda)$ curve takes ever more $\Pi$-shape form. So, it would be natural to agree to the following definition.

Definition 1.1 Mirror high-reflectance zones are wavelength regions, where the transmittance tends to zero as the number of periods tends to infinity. The zone on the extreme right of the wavelength axis is perceived as the principal high-reflectance zone.

Let us consider the Chebyshev polynomial argument in the expression for the system characteristic matrix. It depends on the wavelength of the incident light (being connected through the wavenumber $k$ ) and on the incidence angle $\gamma_{a}$ (through the $\gamma_{1}$ and $\gamma_{2}$ angles connected with $\gamma_{a}$ by the Snell law). So further we will designate it as $a\left(\lambda, \gamma_{a}\right)$.

Employing the Chebyshev polynomial properties, we can show that high-reflectance zones are determined by the condition

$$
\begin{equation*}
\left|a\left(\lambda, \gamma_{a}\right)\right|>1 . \tag{1.5.18}
\end{equation*}
$$

We will use this inequality as the basis for further analysis.
First of all, let us investigate the position of high-reflectance zones at normal light incidence. This condition observed, $\varphi_{1}$ and $\varphi_{2}$ in the expression for $a$ coincide and are equal to

$$
\varphi=k n_{1} d_{1}=k n_{2} d_{2}=\frac{2 \pi}{\lambda} \frac{\lambda_{0}}{4}=\frac{\pi \lambda_{0}}{2 \lambda},
$$

$q_{1,2}$ parameters are equal to corresponding refractive indices. So

$$
\begin{equation*}
a(\lambda, 0)=\cos ^{2}\left(\frac{\pi \lambda_{0}}{2 \lambda}\right)-\frac{1}{2}\left(\frac{n_{1}}{n_{2}}+\frac{n_{2}}{n_{1}}\right) \sin ^{2}\left(\frac{\pi \lambda_{0}}{2 \lambda}\right) \tag{1.5.19}
\end{equation*}
$$

It follows from (1.5.19) that $a(\lambda, 0)$ cannot be more than a unit. Hence, the condition of a $a(\lambda, 0)<-1$ must be observed in the high-reflectance zone, whose boundaries will be defined from the equation

$$
\begin{equation*}
a(\lambda, 0)=-1 \tag{1.5.20}
\end{equation*}
$$

Substituting (1.5.19) into (1.5.20) and expressing sine and cosine squares through a double angle cosine, we will obtain the following equation to determine the high-reflectance zone boundaries:

$$
\begin{equation*}
\cos \frac{\pi \lambda_{0}}{\lambda}=\frac{n_{1}^{2}+n_{2}^{2}-6 n_{1} n_{2}}{\left(n_{1}+n_{2}\right)^{2}} \tag{1.5.21}
\end{equation*}
$$

Let us designate the value in the right part of (1.5.21) as $\xi$. It is easy to show that $\xi>-1$ ( $\xi=-1$ only at $n_{1}=n_{2}$ ) and grows monotonously with a growing ratio of high and low refractive indices. For practically all the materials employed $\xi<0$. The solution of equation (1.5.21) can be conveniently represented in the $\pi \lambda_{0} / \lambda$ argument trigonometry circle (see Fig. 1.11). A bold solid line marks a part of the circle where condition $a(\lambda, 0)<-1$ is satisfied.

Let us designate the upper and the lower boundaries of the high-reflectance zones at normal light incidence as $\lambda_{u}$ and $\lambda_{d}$, respectively. As follows from Fig. 1.11, they are determined from the formulas

$$
\begin{align*}
& \lambda_{u}=\lambda_{0} \frac{\pi}{\pi(2 m+1)-\arccos (-\xi)}  \tag{1.5.22}\\
& \lambda_{d}=\lambda_{0} \frac{\pi}{\pi(2 m+1)+\arccos (-\xi)}
\end{align*}
$$

where $\xi=\frac{n_{1}^{2}+n_{2}^{2}-6 n_{1} n_{2}}{\left(n_{1}+n_{2}\right)^{2}}$.
Formulas (1.5.22) do not depend on the order of high and low refractive index alteration. So, $n_{1}, n_{2}$ in the expression for $\xi$ can be altered for $n_{H}, n_{L}$.

These formulas describe the boundaries of all high-reflectance zones. For the principal high-reflectance zone $m$ equals to zero.

With $m=1,2, \ldots$ formulas (1.5.22) determine the boundaries of the adjacent highreflectance zones, located on the wavelength axis to the left of the principal one. The width of the adjacent zones decreases fast as $m$ grows, and usually, only the first one of them is of any practical importance. A relative width of the principal zone is determined by the ratio

$$
\begin{equation*}
\frac{\lambda_{u}}{\lambda_{d}}=\frac{\pi+\arccos (-\xi)}{\pi-\arccos (-\xi)} \tag{1.5.23}
\end{equation*}
$$



Figure 1.11: On the determination of the high-reflectance zone boundaries

Note, that, according to (1.5.22), (1.5.23), both the relative width and the ratio of the boundary wavelengths to the central one depend only on the $n_{H} / n_{L}$ ratio. Fig. 1.12 shows graphic dependence of $\lambda_{u} / \lambda_{d}, \lambda_{u} / \lambda_{0}$ and $\lambda_{d} / \lambda_{0}$ on the ratio of the refractive indices within the range of $n_{H} / n_{L} \quad$ changes from 1 to 3 .


Figure. 1.12: Correlation between the relative width of the principal high-reflectance zone and the ratio of high and low refractive indices

Now, let us consider how the positions of high-reflectance zones change with incidence deviations from the normal.

For every value of the $\gamma_{a}$ the boundary wavelengths are found from the equation

$$
\begin{equation*}
a\left(\lambda, \gamma_{a}\right)=-1 \tag{1.5.24}
\end{equation*}
$$

Thus, equation (1.5.24) determines the boundary wavelengths as implicit functions of $\gamma_{a}$.
Omitting the intermediate stages, let us pass to the final expressions for the boundaries deviations from values at the normal incidence. Let us designate the upper and the lower boundary deviations as $\Delta \lambda_{u}$ and $\Delta \lambda_{d}$, respectively. The formulas below are obtained in approximation of small incidence deviations from the normal. They are valid for all highreflectance zones of the mirror. We supply them simultaneously for the field $S$ - and $P$-components:

$$
\begin{equation*}
\frac{\Delta \lambda_{u}}{\lambda_{u}}=\left[-\frac{1}{4}\left(\frac{1}{n_{H}^{2}}+\frac{1}{n_{L}^{2}}\right) \pm \frac{\lambda_{u}}{\pi \lambda_{0}} \frac{\sqrt{n_{H} n_{L}}\left(n_{H}-n_{L}\right)}{n_{H}^{2} n_{L}^{2}}\right] n_{a}^{2} \gamma_{a}^{2} \tag{1.5.25}
\end{equation*}
$$

(the positive sign before the second term in the square brackets corresponds to the $S$-case, while the negative sign stands for the $P$-case),

$$
\begin{equation*}
\frac{\Delta \lambda_{d}}{\lambda_{d}}=\left[-\frac{1}{4}\left(\frac{1}{n_{H}^{2}}+\frac{1}{n_{L}^{2}}\right) \mp \frac{\lambda_{d}}{\pi \lambda_{0}} \frac{\sqrt{n_{H} n_{L}}\left(n_{H}-n_{L}\right)}{n_{H}^{2} n_{L}^{2}}\right] n_{a}^{2} \gamma_{a}^{2} \tag{1.5.26}
\end{equation*}
$$

(the positive sign before the second term corresponds to the $P$-case, while the negative sign stands for the $S$-case).

It is easy to show that the first terms in square brackets (1.5.25), (1.5.26) are always greater than the second ones. Hence, $\Delta \lambda_{u, d}$ are always negative. So, high-reflectance zones for both $S$ - and $P$-waves shift into the short-wave part of the spectrum with the incidence angle deviating from the normal.

In the $S$-case, the value of the upper boundary shift is smaller than that of the lower boundary shift. In the $P$-case, the situation is reverse - a high-reflectance zone becomes more narrow as $\gamma_{a}$ grows.

The high-reflectance zone upper boundaries for the $S$ - and $P$-polarized waves shift into the short-wave part of the spectrum with $\gamma_{a}$ growing at different velocities, featuring a greater velocity for $P$-polarized wave. This effect can be used to design light polarizers. As follows from (1.5.25), (1.5.26) the effect augments as the ratio of high and low refractive indices grows.

The formulas (1.5.25), (1.5.26) do not only give a qualitative description of the highreflectance zones positions with the change of the light incidence. Within the range of $30^{\circ}$ incidence, an error in determining the boundary wavelength shifts do not exceed 5-10\% (Tikhonravov, 1983).

### 1.5.4 Principal properties of a quarter-wave dielectric mirror in the proximity to the central wavelength

The mirror properties are influenced by refractive indices errors, layer thicknesses errors, incidence angle deviations from the normal. Here we will consider the influences of this factors holding them small. Let us also consider wavelength deviations from the mirror central wavelength small. In the first approximation, the influence of all enumerated factors is interindependent and when investigating one, the others may be neglected as non-existent.

A number of general qualitative conclusions related to the layer parameters errors influence on the mirror properties were obtained and presented in Section 1.4. It was noted, among other things, that thicknesses and refractive indices errors practically do not tell on the
value of the reflectance inside high-reflectance zones. Formulas (1.4.21)-(1.4.25) are significantly simplified for quarter-wave mirrors at the central wavelength. At $\lambda=\lambda_{0}$, the phase thickness of every layer is equal to $\pi / 2$, the phase changes on reflection $\varphi_{r_{1}}$ and $\varphi_{r_{2}}^{\prime}$ are equal to 0 or $\pi$ at the same time (the former is the case of a layer with high refractive index, the latter being the case of a layer with low refractive index). So, in all formulas, $\sin 2 \theta=0, \sin \varphi=1$, $\sin \left(\varphi-\varphi_{r_{1}}\right)=\sin \left(\varphi-\varphi_{r_{2}}^{\prime}\right)= \pm 1, \cos \left(\varphi-\varphi_{r_{1}}\right)=\cos \left(\varphi-\varphi_{r_{2}}^{\prime}\right)=0$. The right part of the expression in formula (1.4.25) turns into 0 , i.e., $T$ and $R$ in the first approximation do not depend on errors in layer thicknesses.

Formula (1.4.21) takes the form:

$$
\begin{equation*}
\Delta R=-\Delta T= \pm 2 \Delta n \frac{T^{2}\left(\sqrt{R_{1}}+\sqrt{R_{2}}\right)\left(1+\sqrt{R_{1} R_{2}}\right)}{T_{1} T_{2}} . \tag{1.5.27}
\end{equation*}
$$

Expressions (1.3.34), (1.3.35) for energy coefficients of the two multilayer subsystem combination are also simplified. Since $\cos 2 \theta=1$ at the central wavelength for all layers, it follows that

$$
\begin{equation*}
T=\frac{T_{1} T_{2}}{\left(1+\sqrt{R_{1} R_{2}}\right)^{2}}, \quad R=\frac{\left(\sqrt{R_{1}}+\sqrt{R_{2}}\right)^{2}}{\left(1+\sqrt{R_{1} R_{2}}\right)^{2}} \tag{1.5.28}
\end{equation*}
$$

These equalities allow to further simplify expression (1.5.27). Taking them into account, we obtain

$$
\begin{equation*}
\Delta R=-\Delta T= \pm 2 \Delta n \frac{T \sqrt{R}}{n} \tag{1.5.29}
\end{equation*}
$$

Formula (1.5.29) shows that in a first approximation the influence of error in refractive index of individual layer on the mirror energy coefficients does not depend on the layer number but is exclusively determined by the ratio $\Delta n / n$. The influence of the error is insignificant, since another small factor $T$ comprises a part of (1.5.29). So, if $T \sim 1 \%$ and the relative error has about the same value of $1 \%$, then $\Delta R \sim 0.02 \%$, which is significantly smaller than the experimental errors of the reflectance measurements.

Formulas (1.3.22)-(1.3.24), connected with a small absorption, can be written down as follows, having in mind (1.5.28)

$$
\begin{align*}
& \Delta T=-\frac{\pi T}{n} \frac{1-\sqrt{R_{1} R_{2}}}{1+\sqrt{R_{1} R_{2}}} \kappa,  \tag{1.5.30}\\
& \Delta R=-\frac{2 \pi R}{n} \frac{\sqrt{R_{1}} T_{2}}{\left(\sqrt{R_{1}}+\sqrt{R_{2}}\right)\left(1+\sqrt{R_{1} R_{2}}\right)} \kappa,  \tag{1.5.31}\\
& A=\frac{\pi T\left(1+R_{1}\right)}{n T_{1}} \kappa . \tag{1.5.32}
\end{align*}
$$

Expression (1.5.30) includes the small factor $T$ besides the small parameter $\kappa$. So the absorption in the layers tells but insignificantly on the transmittance value. Values for $\Delta R$ and $A$, found from (1.5.31), (1.5.32) can be significantly greater, if the $T_{1}$ value like the $T$ value is small, while the $T_{2}$ value is large in comparison with $T$. Since $T_{1}$ is a transmittance of the layers following the one under consideration, and $T_{2}$ is transmittance of the layers preceding it in the mirror, absorption in the outer layers of the mirror exerts greatest influence on the $R$ and $A$ coefficients. Practically the whole of absorption comes to the first 4-6 outer layers.

So, errors in layer thicknesses and refractive indices show an insignificant influence on the energy coefficients of the mirror in the proximity of the central wavelength. At the same time, absorption in the layers has a much greater impact on them. Therefore, below we will consider only the influence of the last of the factors in greater detail.

Let $\tilde{n}_{H}=n_{H}-i \kappa_{H}, \tilde{n}_{L}=n_{L}-i \kappa_{L}$ are complex refractive indices of the mirror layers. The $\kappa_{H}, \kappa_{L}$ are supposedly small. Let us supply the final expression for reflectance of a quarterwave mirror at the central wavelength $\lambda_{0}=4 n_{H} d_{H}=4 n_{L} d_{L}$, (here $d_{H}$ and $d_{L}$ are geometric thicknesses of a mirror layers having, respectively, high and low refractive indices). In case when the outer layer has a high refractive index (Giacomo, 1956)

$$
\begin{equation*}
R=R_{0}-\frac{2 \pi n_{a}\left(\kappa_{H}+\kappa_{L}\right)}{n_{H}^{2}-n_{L}^{2}} . \tag{1.5.33}
\end{equation*}
$$

Here $R_{0}$ is a reflectance at the central wavelength with no absorption in the layers. Remember, that the $R_{0}$ value can be calculated to a high degree of accuracy from approximate formulas (1.5.14), (1.5.17):
if the number of layers in a mirror is even and equal to 2 N

$$
R_{0}=1-4 \frac{n_{a}}{n_{s}}\left(\frac{n_{L}}{n_{H}}\right)^{2 N}
$$

and if the number of layers in a mirror is even and equal to $2 N+1$

$$
R_{0}=1-4 \frac{n_{a} n_{s}}{n_{H}^{2}}\left(\frac{n_{L}}{n_{H}}\right)^{2 N}
$$

In case of a low refractive index of the outer layer

$$
\begin{equation*}
R=R_{0}-\frac{2 \pi\left(\kappa_{L} n_{H}^{2}+\kappa_{H} n_{L}^{2}\right)}{n_{a}\left(n_{H}^{2}-n_{L}^{2}\right)} . \tag{1.5.34}
\end{equation*}
$$

where $R_{0}$ reflectance can be calculated from the approximate formulas (1.5.15), (1.5.16): if the number of layers in a mirror is even and equal to 2 N

$$
R_{0}=1-4 \frac{n_{s}}{n_{a}}\left(\frac{n_{L}}{n_{H}}\right)^{2 N},
$$

and if it is odd and equals $2 N+1$

$$
R_{0}=1-4 \frac{n_{L}^{2}}{n_{a} n_{s}}\left(\frac{n_{L}}{n_{H}}\right)^{2 N}
$$

Expressions (1.5.33), (1.5.34) are obtained as the first approximation with respect to $\kappa_{L}$ and $\kappa_{H}$. Besides, an additional approximation is made in the course of their deduction, similar to those made for formulas (1.5.14)-(1.5.17). Their comparison with precise ones shows that they adequately describe a decrease in the reflectance caused by an absorption in the layers.

Unlike (1.5.30)-(1.5.32), expressions (1.5.33), (1.5.34) describe absorption not in an individual layer of a mirror, but in all the layers at once. Since the customary outer space is air $\left(n_{a}=1\right)$, and $n_{H}, n_{L}>1$ change of $R$ in expression (1.5.34) is significantly greater than that in (1.5.33). It means that mirrors featuring a high refractive index of the outer layer show smaller absorption losses.

Let us now analyze the influence of small deviations of the incidence from the normal and wavelength from the central on the properties of quarter-wave mirrors. Note, that the deviations cause real changes in the $\varphi_{1}$ and $\varphi_{2}$ phase thicknesses and in the $q_{1}, q_{2}$ parameters in expression (1.5.1) for the mirror period matrix. Equivalent real changes in the values can be caused by layer thicknesses and refractive indices variations. But, as was shown at the beginning of this section, $n$ and $d$ variations tell but insignificantly on the values of mirror energy
coefficients. So, the influence of the incidence angle and the wavelength deviations on mirror energy coefficients, will also be insignificant.

However, these deviations may exert a noticeable influence on the phase change on reflection. Mirror phase characteristics are also of great importance, in particular, in connection with investigating interference filters properties. So, let us supply the formula (Seeley, 1964; Ishiguro and Kato, 1953), describing the value of the phase change $\varphi_{r}$ at small $\gamma_{a}$ and small deviations of wavelengths $\Delta \lambda=\lambda-\lambda_{0}$.

If the outer layer has a high refractive index, then

$$
\begin{equation*}
\varphi_{r}=\pi+\pi \frac{n_{a}}{n_{H}-n_{L}} \frac{\Delta \lambda}{\lambda_{0}}+\frac{\pi}{2} \frac{n_{a}^{3}}{n_{H} n_{L}\left(n_{H}-n_{L}\right)} \gamma_{a}^{2} . \tag{1.5.35}
\end{equation*}
$$

It is outer layer has a low refractive index, then

$$
\begin{equation*}
\varphi_{r}=\pi \frac{n_{H} n_{L}}{n_{a}\left(n_{H}-n_{L}\right)} \frac{\Delta \lambda}{\lambda_{0}}+\frac{\pi}{2} \frac{n_{a}\left(n_{H}^{2}-n_{H} n_{L}+n_{L}^{2}\right)}{n_{H} n_{L}\left(n_{H}-n_{L}\right)} \gamma_{a}^{2} . \tag{1.5.36}
\end{equation*}
$$

We supply these formulas without deduction process since it is time- and spaceconsuming.

Let us only make one remark on the influence of absorption on the phase properties of mirrors. We can show that in all cases in the expression for $\varphi_{r}$ there will also be small factors of the $T$ order in all terms containing $\kappa_{H}, \kappa_{L}$. Due to this, absorption is insignificant for the value of the phase change on reflection.

### 1.5.5 Dielectric mirrors on a metal substrate

Here we will obtain simple analytical formulas describing major properties of dielectric mirrors on a metal substrate. For this purpose let us consider, in the most general form, a system consisting of a multilayer dielectric coating applied onto a surface with $n_{s}-i \kappa_{s}$ complex refractive index.

It is convenient to express the system as a combination of two subsystems, the first one being the boundary between media featuring $n_{s}-i \kappa_{s}$ and $n_{a}$ refractive indices; and the other one a dielectric coating limited on both sides by media with $n_{a}$ refractive indices. Then formula
(1.3.32) with $\varphi=0$ can be used in order to find the amplitude reflectance of the metal substrate mirror:

$$
\begin{equation*}
r=\frac{r_{2}+r_{1}\left(t_{2} t_{2}^{\prime}-r_{2} r_{2}^{\prime}\right)}{1-r_{1} r_{2}^{\prime}} . \tag{1.5.37}
\end{equation*}
$$

Here, $r_{1}$ is an amplitude reflectance of the first subsystem, $t_{2}, r_{2}, t_{2}^{\prime}, r_{2}^{\prime}$ are amplitude coefficients of the dielectric coating in the direct and reverse (from the substrate to the outer space) directions.

Amplitude reflectance $r_{1}$ is equal to

$$
r_{1}=\frac{n_{a}-n_{s}+i \kappa_{s}}{n_{a}+n_{s}-i \kappa_{s}},
$$

and reflectance $R_{1}=\left|r_{1}\right|^{2}$ is

$$
\begin{equation*}
R_{1}=\frac{\left(n_{a}-n_{s}\right)^{2}+\kappa_{s}^{2}}{\left(n_{a}+n_{s}\right)^{2}+\kappa_{s}^{2}}, \tag{1.5.38}
\end{equation*}
$$

Let us denote the phase shift on the reflection from the boundary with the substrate as $\varphi_{1}$ :

$$
\begin{equation*}
\varphi_{1}=\arg r_{1}=\arctan \frac{\kappa_{s}}{n_{a}-n_{s}}+\arctan \frac{\kappa_{s}}{n_{a}+n_{s}}=\pi-\arctan \frac{2 \kappa_{s} n_{a}}{\kappa_{s}^{2}+n_{s}^{2}-n_{a}^{2}} \tag{1.5.39}
\end{equation*}
$$

Assume first that a dielectric coating is non-absorbing. Then the reflectance from both sides of the coating is identical: $\left|r_{2}\right|^{2}=\left|r_{2}^{\prime}\right|^{2}=R_{2}$. Let us assume $\varphi_{2}$ and $\varphi_{2}^{\prime}$ as phase shifts on the reflections from different sides of a coating: $\varphi_{2}=\operatorname{agr} r_{2}, \varphi_{2}^{\prime}=\operatorname{agr} r_{2}^{\prime}$. Assume also $\psi_{2}=\operatorname{agr} t_{2}, \psi_{2}^{\prime}=\operatorname{agr} t_{2}^{\prime}$. When there is no absorption, the formula (see Section 1.3) is valid:

$$
\begin{equation*}
\psi_{2}=\psi^{\prime}, \quad 2 \psi_{2}=\pi-\varphi_{2}-\varphi_{2}^{\prime} . \tag{1.5.40}
\end{equation*}
$$

Using the designations introduced in (1.5.40), we can put down expression (1.5.37) as follows:

$$
\begin{equation*}
r=\frac{\sqrt{R_{2}}-\sqrt{R_{1}} e^{i\left(\varphi_{1}+\varphi_{2}^{\prime}\right)}}{1-\sqrt{R_{1} R_{2}} e^{i\left(\varphi_{1}+\varphi_{2}^{\prime}\right)}} e^{i \varphi_{2}} \tag{1.5.41}
\end{equation*}
$$

For the reflectance $R=|r|^{2}$ we obtain

$$
R=\frac{R_{1}+R_{2}-2 \sqrt{R_{1} R_{2}} \cos \left(\varphi_{1}+\varphi_{2}^{\prime}\right)}{1+R_{1} R_{2}-2 \sqrt{R_{1} R_{2}} \cos \left(\varphi_{1}+\varphi_{2}^{\prime}\right)},
$$

which can be conveniently rewritten as

$$
\begin{equation*}
R=1-\frac{\left(1-R_{1}\right)\left(1-R_{2}\right)}{1+R_{1} R_{2}-2 \sqrt{R_{1} R_{2}} \cos \left(\varphi_{1}+\varphi_{2}^{\prime}\right)} \tag{1.5.42}
\end{equation*}
$$

Let us study the $R$ coefficient in the proximity of the central wavelength of the quarterwave mirror. Assume for the sake of simplicity, that in the proximity of $\lambda_{0}$, the dispersion of the complex refractive index of the metal is small, and $R_{1}, \varphi_{1}$ may be regarded as constants. Since metals usually have $\kappa_{s}$, significantly larger that $n_{s}$, and $n_{a}$, the value of $R_{1}$ is close to 1 , and $\varphi_{1}$ close to $\pi$ (see (1.5.38), (1.5.39)). As was shown in the previous section, in the first approximation the phase shift on the reflection $\varphi_{2}^{\prime}$ grows linearly in the proximity of $\lambda_{0}$, and the $R_{2}$ value does not change. So, near the central wavelength in (1.5.42) only $\cos \left(\varphi_{1}+\varphi_{2}^{\prime}\right)$ change in the first approximation. Note, that the maximal value of R in (1.5.42) is obtained at $\cos \left(\varphi_{1}+\varphi_{2}^{\prime}\right)=1$, while the minimal value at $\cos \left(\varphi_{1}+\varphi_{2}^{\prime}\right)=-1$. Depending on the structure of a dielectric mirror in the proximity of $\lambda_{0}$ one of these conditions is fulfilled.

If the metal substrate is adjacent to the layer with low refractive index, the $\varphi_{2}^{\prime}$ value is determined from the formula

$$
\varphi_{2}^{\prime}=\pi \frac{n_{H} n_{L}}{n_{a}\left(n_{H}-n_{L}\right)}\left(\frac{\lambda}{\lambda_{0}}-1\right) .
$$

At

$$
\begin{equation*}
\lambda=\lambda_{0}\left[1+\frac{n_{a}\left(n_{H}-n_{L}\right)}{\pi n_{H} n_{L}} \arctan \frac{2 \kappa_{s} n_{a}}{\kappa_{s}^{2}+n_{s}^{2}-n_{a}^{2}}\right] \tag{1.5.43}
\end{equation*}
$$

the sum $\varphi_{1}+\varphi^{\prime}$ is equal to $\pi$ and $\cos \left(\varphi_{1}+\varphi_{2}^{\prime}\right)=-1$. The maximal value of $R$ equals to:

$$
R_{\max }=1-\frac{\left(1-R_{1}\right)\left(1-R_{2}\right)}{\left(1+\sqrt{R_{1} R_{2}}\right)^{2}}
$$

This value can be very high. Let, for instance, $R_{1}=R_{2}=0.9$. Then $R_{\max }>0.997$. The reflectance value all over high reflection zone is almost as high. So, a two-component system of quarter-wave dielectric layers on a metal substrate is a good reflector. To achieve a high reflectance of a mirror on a metal substrate, it requires significantly fewer layers than a usual quarter-wave mirror on a dielectric substrate has. Note, that according to (1.5.43), the wavelength giving a reflectance maximum is shifted in the direction of large values with respect to the central wavelength of quarter-wave layers.

Let now, a layer with a high refractive index be adjacent to a metal substrate. Then the phase shift $\varphi_{2}^{\prime}$ is found from the formula

$$
\varphi_{2}^{\prime}=\pi+\pi \frac{n_{a}}{n_{H}-n_{L}}\left(\frac{\lambda}{\lambda_{0}}-1\right)
$$

The sum of $\varphi_{1}$ and $\varphi_{2}^{\prime}$ in the proximity of $\lambda_{0}$ is close to $2 \pi$. At wavelength

$$
\begin{equation*}
\lambda=\lambda_{0}\left[1+\frac{n_{H}-n_{L}}{\pi n_{a}} \arctan \frac{2 \kappa_{s} n_{a}}{\kappa_{s}^{2}+n_{s}^{2}-n_{a}^{2}}\right] \tag{1.5.44}
\end{equation*}
$$

the following equalities are valid: $\varphi_{1}+\varphi_{2}^{\prime}=2 \pi$ and $\cos \left(\varphi_{1}+\varphi_{2}^{\prime}\right)=1$. At this value of $\lambda$, the reflectance is minimal and equal to

$$
R_{\min }=\left(\frac{\sqrt{R_{1}}-\sqrt{R_{2}}}{1-\sqrt{R_{1} R_{2}}}\right)^{2}
$$

The $R_{\min }$ value may be very small. Thus, when $R_{1}=R_{2}$, it turns into 0 . It means that in the proximity of $\lambda_{0}$ there may exist deep gaps in the reflectance curve. It is highly undesirable for a mirror. Hence, a conclusion follows that mirrors on a metal substrate should not have a high refractive index in the first layer from the substrate.

Now, let us consider the influence of small absorption in dielectric layers on the mirror reflectance.

We can show that the change of the energy coefficient $R$ in the area of reflectance maximum is determined from the approximate formula

$$
\begin{equation*}
R=R_{\max }-p(k)=1-\frac{\left(1-R_{1}\right)\left(1-R_{2}\right)}{\left(1+\sqrt{R_{1} R_{2}}\right)^{2}}-p(k) \tag{1.5.45}
\end{equation*}
$$

where
if the outer layer has a high refractive index

$$
p(k)=\frac{2 \pi n_{a}\left(\kappa_{H}+\kappa_{L}\right)}{n_{H}^{2}-n_{L}^{2}},
$$

and if the outer layer has a low refractive index

$$
p(k)=\frac{2 \pi\left(\kappa_{L} n_{H}^{2}+\kappa_{H} n_{L}^{2}\right)}{n_{a}\left(n_{H}^{2}-n_{L}^{2}\right)} .
$$

Thus, absorption in dielectric layers of a quarter-wave mirror on a metal substrate results in a similar decrease of reflectance as for mirrors on dielectric substrates. Note, that formula (1.5.45) gives values for $R$ which are close to those calculated from precise recurrent formulas.

The reflectance decrease in mirrors with high refractive index of the outer layer is smaller. It was shown above that the substrate should be adjacent to a low refractive index layer. Thus, quarter-wave mirrors on a metal substrate must have an even number of layers. Theoretically, the $R_{\max }$ value is close to $100 \%$ in case of 6-8 layers of coating. However, as formula (1.5.45) shows, the losses in dielectric layers are an obstacle on the way of obtaining super high reflectance.

### 1.6 Approximate formulas for dielectric narrow bandpass filters

This section does not consider the whole of the diversity of dielectric narrow bandpass filters. A detailed description of various designs of narrow bandpass filters can be found in A.Thelen's book (1989). Here we analyze the influence of absorption and deviation of the incidence angle on the spectral properties of the filters in the proximity of their central wavelength. The dependence of the filter properties on the errors in the refractive indices and layer thicknesses is not considered here as the influence of the factors can be to a great extent overcome due to a selection of the monitoring method for coating deposition (Furman, 1977). In more detailed way many questions connected with investigating dielectric narrow bandpass filters are considered in the Sh.A.Furman book (1977).

### 1.6.1 Common expressions for spectral coefficients of filter-type systems near central wavelength

It is sensible to begin the analysis with the deduction of common formulas for transmittance, reflectance and absorptance of special type multilayer systems to be further referred to as ''filter-type systems".

A standard narrow bandpass filter is a multilayer system consisting of two quarter-wave mirrors with a half-wave layer placed between them. The phase changes on reflection from mirrors at the central wavelength are both equal to either zero or $\pi$, the double phase thickness of the spacer layer equals $2 \pi$. Thus, the $\theta$ value in (1.3.34), (1.3.35) at the filter central wavelength is divisible by $\pi$ in all cases.

We made this remark for the purpose of clarifying from the very start the phase condition concept that we introduce for filter-type systems. Let us now outline systems to be designated by this term. Let us study the system in Fig. 1.9, consisting of two multilayer subsystems with a spacer layer between them. Below, we will use designations introduced in subsection 1.3.3. Suppose, the subsystem reflectances are close to a unit, and that they, like the spacer layer, are slightly absorbing. Further they will be referred to as mirrors 1 and 2, respectively. Let us denote the multilayer system in question as a filter-type system if the following condition is satisfied at a certain wavelength $\lambda_{0}$ and the normal incidence: the $\theta$ value calculated without an absorption taken into account, is divisible by $\pi$. We will also call the wavelength $\lambda_{0}$ central.

The following two formulas from subsection 1.3 .3 will be used as initial ones for the filter-type system research:

$$
\begin{align*}
& \frac{1}{t}=\frac{e^{i \varphi}-r_{1} r_{2}^{\prime} e^{-i \varphi}}{t_{1} t_{2}}  \tag{1.6.1}\\
& \frac{r}{t}=\frac{r_{2} e^{i \varphi}+r_{1}\left(t_{2} t_{2}^{\prime}-r_{2} r_{2}^{\prime}\right) e^{-i \varphi}}{t_{1} t_{2}} \tag{1.6.2}
\end{align*}
$$

We will investigate the system properties in the proximity of the central wavelength at the incidence close to the normal. It is evident from the above that the senior terms connected with the incidence angle, will be of the $\gamma_{a}^{2}$ order. Let $\kappa$ be the imaginary part of the spacer layer refractive index. We supposed it be a small value. Let us consider the parameters characterizing the absorption in mirrors 1 and 2 to be values of the same order as $\kappa$. Later we will introduce the concrete values of these parameters, while for the time being, the supposition is sufficient to obtain the main formulas, describing the properties of the filter-type systems. All the calculations will be carried out with an accuracy to the first order of $\Delta \lambda=\lambda-\lambda_{0}, \gamma_{a}^{2}$ and $\kappa$. Besides, we will neglect the terms containing small factors of the mirror transmittances alongside with the small parameters.

The mirrors characteristics at the central wavelength, normal incidence and in nonabsorbing case will be marked with the 0 upper index. Assume,

$$
\varphi^{0}=\frac{2 \pi}{\lambda_{0}} n d
$$

where $n$ is a real part of the spacer layer refractive index, and $d$ is its thickness.
Let us put down the amplitude reflectances of the mirrors as

$$
\begin{aligned}
& r_{1}=\sqrt{R_{1}^{0}-\alpha_{1}} e^{i\left(\varphi_{1}^{0}+\beta_{1}\right)}, \\
& r_{2}=\sqrt{R_{2}^{0}-\alpha_{2}} e^{i\left(\varphi_{2}^{0}+\beta_{2}\right)}, \\
& r_{2}^{\prime}=\sqrt{R_{2}^{0}-\alpha_{2}^{\prime}} e^{i\left(\varphi_{2}^{0}+\beta_{2}^{\prime}\right)} .
\end{aligned}
$$

Here we designated all the terms of the first order of smallness with respect to $\Delta \lambda, \gamma_{a}^{2}, \kappa$ taking account of the influence of the corresponding factors as $\alpha_{1}, \alpha_{2}, \alpha_{2}^{\prime}$ and $\beta_{1}, \beta_{2}, \beta_{2}^{\prime}$. Since the amplitude transmittances are small, we will neglect the above factors influence, and assume

$$
\begin{aligned}
& t_{1}=\sqrt{\frac{n}{n_{s}} T_{1}^{0}} e^{i \psi_{1}^{0}}, \\
& t_{2}=\sqrt{\frac{n_{a}}{n} T_{2}^{0}} e^{i \psi_{2}^{0}} \\
& t_{2}^{\prime}=\sqrt{\frac{n}{n_{a}} T_{2}^{0}} e^{i \frac{i 2_{2}^{0}}{2}}
\end{aligned}
$$

It has been taken into account here that the phase changes on transmission in the direct and reverse directions are identical. The following equations have also been taken into account:

$$
T_{1}^{0}=\frac{n_{s}}{n}\left|t_{1}^{0}\right|^{2}, \quad T_{2}^{0}=T_{2}^{\prime 0}=\frac{n}{n_{a}}\left|t_{2}^{0}\right|^{2}=\frac{n_{a}}{n}\left|t_{2}^{0}\right|^{2}
$$

Taking account of the energy equation $R_{1}^{0}=1-T_{1}^{0}$, the smallness of the transmittance $T_{1}^{0}$ and terms $\alpha_{1}, \beta_{2}$ we may re-write the expression for the $r_{1}$ amplitude reflectance in the following approximate form:

$$
r_{1} \cong\left(1-\frac{T_{1}^{0}}{2}-\frac{\alpha_{1}}{2}+i \beta_{1}\right) e^{i \varphi_{1}^{0}} .
$$

Similarly, we will put down the remaining amplitude reflectances. Designating as $\Delta \varphi$ a change in the spacer layer phase thickness connected with the wavelength deviation from the central one, the deviation of the incidence angle from the normal and absorption taken account of, we will have the following:

$$
\begin{equation*}
\Delta \varphi=-\varphi^{0}\left(\frac{\Delta \lambda}{\lambda_{0}}+\frac{n_{a}^{2} \gamma_{a}^{2}}{2 n^{2}}\right)-i \varphi^{0} \frac{\kappa}{n} . \tag{1.6.3}
\end{equation*}
$$

At small values of $\Delta \varphi$

$$
\begin{aligned}
& e^{i \varphi}=e^{i\left(\varphi_{0}+\Delta \varphi\right)} \cong(1+i \Delta \varphi) e^{i \varphi^{0}}, \\
& e^{-i \varphi} \cong(1-i \Delta \varphi) e^{-i \varphi^{0}} .
\end{aligned}
$$

Let us substitute all the approximate equalities into (1.6.1) and (1.6.2), bearing in mind the phase relation (1.3.26) for mirror 2:

$$
\varphi_{2}^{\prime}=\pi-\varphi_{2}^{0}+2 \psi_{2}^{0}
$$

as well as the fact that according to filter-type system definition, the value of $\varphi_{1}^{0}+\varphi_{2}^{\prime 0}-2 \varphi^{0}$ is divisible by $2 \pi$.

As a result, we obtain the following:

$$
\begin{aligned}
& \frac{1}{t}=\sqrt{\frac{n_{s}}{n_{a} T_{1}^{0} T_{2}^{0}}} e^{i \varphi^{0}}\left\{\frac{T_{1}^{0}+T_{2}^{0}+\alpha_{1}+\alpha_{2}^{\prime}}{2}+2 i \Delta \varphi-i\left(\beta_{1}+\beta_{2}^{\prime}\right)\right\}, \\
& \frac{r}{t}=\sqrt{\frac{n_{s}}{n_{a} T_{1}^{0} T_{2}^{0}}} e^{i\left(\varphi^{0}+\varphi_{2}^{0}\right)}\left\{\frac{T_{1}^{0}-T_{2}^{0}+\alpha_{1}+\alpha_{2}^{\prime}}{2}+2 i \Delta \varphi-i\left(\beta_{1}+\beta_{2}^{\prime}\right)\right\} .
\end{aligned}
$$

Let us multiply the square modules of the expressions by $n_{a} / n_{s}$. Here we obtain two relations for filter-type system energy coefficients:

$$
\begin{align*}
& \frac{1}{T}=\frac{1}{4 T_{1}^{0} T_{2}^{0}}\left\{\left(T_{1}^{0}+T_{2}^{0}+\alpha_{1}+\alpha_{2}^{\prime}-4 \operatorname{Im} \Delta \varphi\right)^{2}+\left(2 \beta_{1}+2 \beta_{2}^{\prime}-4 \operatorname{Re} \Delta \varphi\right)^{2}\right\},  \tag{1.6.4}\\
& \frac{R}{T}=\frac{1}{4 T_{1}^{0} T_{2}^{0}}\left\{\left(T_{1}^{0}-T_{2}^{0}+\alpha_{1}+\alpha_{2}^{\prime}-4 \operatorname{Im} \Delta \varphi\right)^{2}+\left(2 \beta_{1}+2 \beta_{2}^{\prime}-4 \operatorname{Re} \Delta \varphi\right)^{2}\right\} .
\end{align*}
$$

Introducing the following designations:

$$
\begin{align*}
& c=\frac{T_{1}^{0}}{T_{2}^{0}} \\
& x=\frac{\alpha_{1}+\alpha^{\prime}-4 \operatorname{Im} \Delta \varphi}{T_{2}^{0}},  \tag{1.6.5}\\
& y=\frac{2 \beta_{1}+2 \beta_{2}^{\prime}-4 \operatorname{Re} \Delta \varphi}{T_{2}^{0}},
\end{align*}
$$

and applying them, we will obtain from (1.6.4) that:

$$
\begin{align*}
& T=\frac{4 c}{(1+c+x)^{2}+y^{2}}  \tag{1.6.6}\\
& R=\frac{(1-c-x)^{2}+y^{2}}{(1+c+x)^{2}+y^{2}} . \tag{1.6.7}
\end{align*}
$$

We also obtain from the energy equation $A=1-R-T$ that:

$$
\begin{equation*}
A=\frac{4 x}{(1+c+x)^{2}+y^{2}} . \tag{1.6.8}
\end{equation*}
$$

The obtained expressions simplify the study of principal properties of various narrow bandpass filters. In the following subsection we will apply them for the analysis of two-material dielectric filters, consisting of two quarter-wave mirrors and a half-wave spacer layer.

### 1.6.2 Research of narrow bandpass filters with a half-wave central layer

In the subsection we will obtain simple approximate expressions for the filter transmittance maximum, its half-width at the $0.5 T_{\max }$ level, transmittance maximum wavelength shift with the incidence deviation from the normal.

If the spacer (central) layer has a low refractive index, the adjacent outer layers of the mirrors have a high refractive index. Here we obtain from (1.5.33) and (1.5.35) that

$$
\begin{aligned}
& \alpha_{1}=\alpha_{2}^{\prime}=2 \pi \frac{n_{L}\left(\kappa_{H}+\kappa_{L}\right)}{n_{H}^{2}-n_{L}^{2}} \\
& \beta_{1}=\beta_{2}^{\prime}=\pi \frac{n_{L}}{n_{H}-n_{L}} \frac{\lambda-\lambda_{0}}{\lambda_{0}}+\frac{\pi}{2} \frac{n_{L}^{2}}{n_{H}\left(n_{H}-n_{L}\right)} \gamma^{2}
\end{aligned}
$$

where the $\gamma$ angle is connected with the $\gamma_{0}$ incidence angle by the equation $n_{L} \gamma=n_{a} \gamma_{a}$. In case when the spacer layer has a high index the adjacent mirror layers have a low refractive index, and it follows from (1.5.34) and (1.5.36) that

$$
\begin{aligned}
& \alpha_{1}=\alpha_{2}^{\prime}=2 \pi \frac{\kappa_{L} n_{H}^{2}+\kappa_{H} n_{L}^{2}}{n_{H}\left(n_{H}^{2}-n_{L}^{2}\right)} \\
& \beta_{1}=\beta_{2}^{\prime}=\pi \frac{n_{L}}{n_{H}-n_{L}} \frac{\lambda-\lambda_{0}}{\lambda_{0}}+\frac{\pi}{2} \frac{n_{H}^{2}-n_{H} n_{L}+n_{L}^{2}}{n_{L}\left(n_{H}-n_{L}\right)} \gamma^{2},
\end{aligned}
$$

where the $\gamma$ angle is determined through $\gamma_{a}$ from the equation $n_{H} \gamma=n_{a} \gamma_{a}$.
The phase thickness $\varphi^{0}$ in (1.6.3) is equal to $\pi$, since $n d=\lambda_{0} / 2$. Substituting concrete values into (1.6.5), we obtain for the low refractive index of the central layer

$$
\begin{align*}
& x=\frac{4 \pi\left(n_{H}^{2} \kappa_{L}+n_{L}^{2} \kappa_{H}\right)}{T_{2}^{0} n_{L}\left(n_{H}^{2}-n_{L}^{2}\right)},  \tag{1.6.9}\\
& y=\frac{4 \pi}{T_{2}^{0}}\left[\frac{n_{L}}{n_{H}-n_{L}} \frac{\lambda-\lambda_{0}}{\lambda_{0}}+\frac{n_{H}^{2}-n_{H} n_{L}+n_{L}^{2}}{2 n_{L}^{2} n_{H}\left(n_{H}-n_{L}\right)} n_{a}^{2} \gamma_{a}^{2}\right] .
\end{align*}
$$

In the case of the high refractive index of the central layer, we have

$$
\begin{align*}
& x=\frac{4 \pi}{T_{2}^{0}} \frac{n_{H}\left(\kappa_{H}+\kappa_{L}\right)}{n_{H}^{2}-n_{L}^{2}}, \\
& y=\frac{4 \pi}{T_{2}^{0}}\left[\frac{n_{H}}{n_{H}-n_{L}} \frac{\lambda-\lambda_{0}}{\lambda_{0}}+\frac{1}{2 n_{L}\left(n_{H}-n_{L}\right)} n_{a}^{2} \gamma_{a}^{2}\right] . \tag{1.6.10}
\end{align*}
$$

As follows from (1.6.9), (1.6.10) extinction coefficients enter the formulas (1.6.6)- (1.6.8) only through $x$, while the wavelength and incidence angle are only manifested through $y$.

It follows from formula (1.6.6) that the transmittance maximum is attained at $y=0$. It equals

$$
\begin{equation*}
T_{\max }=\frac{4 c}{(1+c+x)^{2}} \tag{1.6.11}
\end{equation*}
$$

At the normal incidence, the value of $y=0$ is attained at the central wavelength $\lambda_{0}$. With the incidence deviation from the normal the filter transmittance maximum is shifted toward shorter wavelengths. The shift value (designated as $\delta \lambda$ is determined from the condition $y=0$. We obtain from (1.6.9) for the central layer low refractive index

$$
\begin{equation*}
\frac{\delta \lambda}{\lambda}=-\frac{\left(n_{H}^{2}-n_{H} n_{L}+n_{L}^{2}\right) n_{a}^{2}}{2 n_{H}^{2} n_{L}^{2}} \gamma_{a}^{2}, \tag{1.6.12}
\end{equation*}
$$

while (1.6.10) gives for the central layer high refractive index

$$
\begin{equation*}
\frac{\delta \lambda}{\lambda}=-\frac{n_{a}^{2}}{2 n_{H} n_{L}} \gamma_{a}^{2} . \tag{1.6.13}
\end{equation*}
$$

The shape of the transmittance curve does not change.
In the case of no-absorption $(x=0)$ and normal incidence, the transmittance maximum equals

$$
\begin{equation*}
T_{\max }^{0}=\frac{4 c}{(1+k)^{2}} \tag{1.6.14}
\end{equation*}
$$

The $c$-value is a ratio of the filter front and back mirrors transmittances. If the mirrors have the same number of layers, then, depending on the order of layers with high and low refractive indices it equals $n_{a} / n_{s}$ or $n_{s} / n_{a}$ (the former stands for a low refractive index of the first from the substrate layer, while the second one represents the high refractive index of the first layer). In both cases,

$$
T_{\max }^{0}=\frac{4 n_{a} n_{s}}{\left(n_{a}+n_{s}\right)^{2}} .
$$

This value coincides with the transmittance of the boundary between the outer space and substrate, and is usually close to a unit. For instance, for $n_{s}=1.52, n_{a}=1.00$ we obtain $T_{\text {max }} \cong 0.96$.

Let us find the half width of the filter at the level of $T=0.5 T_{\max }$ in nonabsorbing case. We designate it as $\Delta \lambda^{0}$. Let the light incidence be normal. Then the half-width is found from equation

$$
\begin{equation*}
y=1+k \tag{1.6.15}
\end{equation*}
$$

where

$$
y=\frac{2 \pi}{T_{2}^{0}} \frac{n_{H}}{n_{H}-n_{L}} \frac{\Delta \lambda^{0}}{\lambda_{0}} .
$$

Let the number of layers in the mirrors be equal to $m$, (consequently, the total number of layers is $2 m+1$ ). In order to find the transmittance $T_{2}^{0}$, we can make use of the approximate formulas from the subsection 1.5.2. Let, first, the number of layers in a mirror be even and a high refractive index layer is adjacent to the substrate. Then $k=n_{s} / n_{a}$, and we have the following approximate expression for $T_{2}^{0}$ :

$$
T_{2}^{0} \cong 4 \frac{n_{a}}{n_{H}}\left(\frac{n_{L}}{n_{H}}\right)^{m} .
$$

Substituting these value in (1.6.15), we find that

$$
\begin{equation*}
\frac{\Delta \lambda^{0}}{\lambda_{0}}=\frac{2\left(n_{a}+n_{s}\right)\left(n_{H}-n_{L}\right)}{\pi n_{H}^{2}}\left(\frac{n_{L}}{n_{H}}\right)^{m} . \tag{1.6.16}
\end{equation*}
$$

Let again the number of layers in a mirror be even, but the layer adjacent to the substrate has a low refractive index. Then $k=n_{a} / n_{s}$ and

$$
T_{2}^{0} \cong 4 \frac{n_{L}}{n_{a}}\left(\frac{n_{L}}{n_{H}}\right)^{m} .
$$

We obtain from (1.6.15) that

$$
\begin{equation*}
\frac{\Delta \lambda^{0}}{\lambda_{0}}=\frac{2\left(n_{a}+n_{s}\right)\left(n_{H}-n_{L}\right)}{\pi n_{a} n_{s}}\left(\frac{n_{L}}{n_{H}}\right)^{m+1} . \tag{1.6.17}
\end{equation*}
$$

It is easy to see that with an odd m the same formulas are obtained: (1.6.16) being the case for high, (1.6.17) being the case for low refractive index of the first layer from the substrate.

Absorption decreases the filter transmittance. Even at small $\kappa_{H}, \kappa_{L}$ the decrease can be significant as the denominator for $x$-expressions includes a small value of $T_{2}^{0}$, the front mirror transmittance. We will obtain a more convenient expression (compared to (1.6.11)) for the filter transmittance maximum in case of absorption. Let us then write down (1.6.11) in the following form

$$
\begin{equation*}
T_{\max }=\frac{4 c}{(1+c)^{2}} /\left(1+\frac{x}{1+c}\right)^{2}=\frac{4 n_{a} n_{s}}{\left(n_{a}+n_{s}\right)^{2}} /\left(1+\frac{x}{1+c}\right)^{2} . \tag{1.6.18}
\end{equation*}
$$

In case of low refractive index of the central layer, we obtain from (1.6.9)

$$
\begin{equation*}
\frac{x}{1+c}=\frac{4 \pi}{T_{2}^{0}(1+c)} \frac{n_{H}^{2} \kappa_{L}+n_{L}^{2} \kappa_{H}}{n_{L}\left(n_{H}^{2}-n_{L}^{2}\right)} . \tag{1.6.19}
\end{equation*}
$$

It follows from (1.6.15) that

$$
\frac{2 \pi}{T_{2}^{0}(1+c)}=\frac{n_{H}-n_{L}}{n_{H}}\left(\frac{\Delta \lambda^{0}}{\lambda_{0}}\right)^{-1} .
$$

Substituting this expression into (1.6.19), and then into (1.6.18) we obtain:

$$
\begin{equation*}
T_{\max }=\frac{4 n_{a} n_{s}}{\left(n_{a}+n_{s}\right)^{2}}\left[1+2 \frac{n_{H}^{2} \kappa_{L}+n_{L}^{2} \kappa_{H}}{n_{H} n_{L}\left(n_{H}+n_{L}\right)}\left(\frac{\Delta \lambda^{0}}{\lambda_{0}}\right)^{-1}\right]^{-2} \tag{1.6.20}
\end{equation*}
$$

In a similar way, but for the case of the filter central layer having a high refractive index, we obtain

$$
\begin{equation*}
T_{\max }=\frac{4 n_{a} n_{s}}{\left(n_{a}+n_{s}\right)^{2}}\left[1+2 \frac{\kappa_{H}+\kappa_{L}}{n_{H}+n_{L}}\left(\frac{\Delta \lambda^{0}}{\lambda_{0}}\right)^{-1}\right]^{-2} . \tag{1.6.21}
\end{equation*}
$$

The filter half-width in case of absorption will also differ from $\Delta \lambda^{0}$. Let us designate it as $\Delta \lambda$. In order to determine $\Delta \lambda$ we have the following instead of (1.6.15):

$$
\begin{equation*}
y=1+c+x \tag{1.6.22}
\end{equation*}
$$

where

$$
y=\frac{2 \pi}{T_{2}^{0}} \frac{n_{H}}{n_{H}-n_{L}} \frac{\Delta \lambda}{\lambda_{0}} .
$$

It follows immediately from (1.6.16) and (1.6.22) that

$$
\begin{equation*}
\frac{\Delta \lambda}{\Delta \lambda^{0}}=1+\frac{x}{1+c} . \tag{1.6.23}
\end{equation*}
$$

Thus, in case of absorption, the filter half-width grows. By way of noncomplicated calculations, we obtain from (1.6.23) for the central layer low refractive index

$$
\begin{equation*}
\frac{\Delta \lambda}{\lambda_{0}}=\frac{\Delta \lambda^{0}}{\lambda_{0}}+2 \frac{n_{H}^{2} \kappa_{L}+n_{L}^{2} \kappa_{H}}{n_{H} n_{L}\left(n_{H}+n_{L}\right)}, \tag{1.6.24}
\end{equation*}
$$

and for the high refractive index of the central layer

$$
\begin{equation*}
\frac{\Delta \lambda}{\lambda_{0}}=\frac{\Delta \lambda^{0}}{\lambda_{0}}+2 \frac{\kappa_{H}+\kappa_{L}}{n_{H}+n_{L}}, \tag{1.6.25}
\end{equation*}
$$

From (1.6.23) and expressions for $T_{\max }$ and $T_{\max }^{0}(1.6 .11),(1.6 .14)$ the following relation between the filter transmittance maximum and its half-width in case of absorption and in nonabsorbing case takes place:

$$
\begin{equation*}
(\Delta \lambda)^{2} T_{\max }=\left(\Delta \lambda^{0}\right)^{2} T_{\max }^{0} \tag{1.6.26}
\end{equation*}
$$

In this section we confined ourselves to considering narrow bandpass dielectric filters featuring exclusively half-wave central layers with the purpose to obtain shorter final expressions. With the help of the relation (1.6.6) it would be also possible to consider filters whose central layer optical thicknesses are equal to $p \lambda_{0} / 2$, with $p$ always being an integer. However, the expressions obtained in this case are more complicated.

Let us note in conclusion that the formulas obtained in this section adequately describe filters properties. This is further proved by the comparison of the values obtained from them and the results of calculations through precise recurrent formulas.

## Chapter 2

## SYNTHESIS OF MULTILAYER OPTICAL COATINGS

The synthesis problems are opposite to those considered in the previous chapter. There we analyzed how, given the parameters of a layered system, to determine its spectral properties and to investigate the impact of various factors. The tasks of the synthesis require the reverse: to determine the parameters of the coating featuring the sought for spectral properties. We should emphasize at once, that solving the synthesis problems we should always keep in mind a subsequent practical implementation of coatings. It means, that seeking for solutions, we should take into consideration the conditions of engineering implementability of the results.

A synthesis by way of analysis is a possible approach to the coating design. When investigating analysis problems, some properties of coatings which can immediately be employed in solving some particular synthesis problems are found. Thus, simple formulas for quarter-wave dielectric mirrors make it possible to determine the number of layers and optical thicknesses providing a pre-set reflectance level at a certain given wavelength.

However, only a few problems can be solved on the basis of analysis, and very frequently, analysis presents a far-from-optimal way of solution. So during the last three decades great significance has been attached to elaborating synthesis methods proper. It should be stated that so far there is no universal synthesis method capable to provide optimal solutions of all the problems arising in practice. We can enumerate up to a dozen methods that have proved their practical significance and have been a valuable and comprehensive contribution to optical coating design. A complete and reliable comparative review of the methods employed was recently suggested by J.A.Dobrowolski (1986).

We believe that at the present stage of development, a rational combination of various methods based mostly on the merit function optimization is most promising. Here we will try to
prove our point of view. With the number of layers in a coating not exceeding 6-8, modern computer methods of merit function optimization enable obtaining a reliable global minimum (optimal design) within reasonable time. When starting designs are used, optimization methods afford good results even with a considerably greater number of layers. Thus, optimization methods have wide potentials of their own. Besides, they are much simpler in mastering and more comprehensive in application. It is for this reason, primarily, that we consider them basic. The major concepts and a number of characteristic peculiarities of the optimization methods are the subject matter of the first section in the chapter.

### 2.3 Synthesis methods based on merit function optimization

Here we consider principal questions related to the design of two-component optical coatings with pre-set alternating refractive indices $n_{H}$ and $n_{L}$. The values sought for in synthesis are the layer thicknesses. As a mathematical investigation (Sveshnikov and Tikhonravov, 1987) shows, optimal solutions for synthesis problems frequently ought to be sought for among this type of coatings. It follows from the results of the research, that materials featuring maximal difference between $n_{H}$ and $n_{L}$ are most promising in designing two-component coatings.

We should note, that the conclusion concerning optimality of two-component coatings may prove invalid when the number of layers is rigidly limited. Thus, a three-layer antireflection coating with three different refractive indices may turn better than a two-component antireflection coating featuring the same number of layers (but it will in all cases be worse than two-component coatings with a bigger number of layers). So, when the number of layers is strictly limited, it is worthwhile to determine their refractive indices. Corresponding methods may be based upon the results described in this section. Let us emphasize, however, that of greatest interest at present are methods for two-component coating design and, moreover, this type of coatings proves to be most convenient technologically.

### 2.1.1 Merit function selection

For the sake of certainty, let us consider the synthesis problem for two-component multilayer dielectric coatings at the normal incidence. The sought for spectral characteristic here will be an energy reflection coefficient. Let us denote the target reflectance dependence as $\tilde{R}(\lambda)$.

Generally speaking there is no coating featuring an exactly required reflectance $\tilde{R}(\lambda)$ So, we can only speak about designing a multilayer coating whose reflectance is in some respect close to $\tilde{R}(\lambda)$.

Let the number of layers equals $m$. We will discuss the question of determining the number of layers later. Since the refractive indices are pre-set, (we also consider their alternation order as pre-set), the values sought for in synthesis are geometrical thicknesses of the layers $d_{1}, \ldots, d_{m}$. Let us introduce an m-dimensional space of the sought for parameters. Its elements are vectors

$$
X=\left\{d_{1}, \ldots, d_{m}\right\}
$$

Let us denote the reflectance of a multilayer coating defined by the $X$ parameter vector, as $R(X, \lambda)$.

Let $\lambda_{d}, \lambda_{u}$ are respectively the lower and the upper boundaries of the spectral band of interest. In order to assess the coating reflectance proximity to the target dependence $\tilde{R}(\lambda)$ let us introduce the merit function

$$
\begin{equation*}
F(X)=\int_{\lambda_{d}}^{\lambda_{u}}[R(X, \lambda)-\tilde{R}(\lambda)]^{2} d \lambda \tag{2.1.1}
\end{equation*}
$$

The merit function in (2.1.1) is known as mean square estimation. It is very computationally convenient primarily because the $F(X)$ function can be differentiated with respect to $X$ (see the next subsection). A disadvantage of the mean square estimation is its low sensitivity to spectrally narrow but big deviations of $R(X, \lambda)$ ) from $\tilde{R}(\lambda)$. However, these deviations often prove to be of little significance from the physical point of view. Besides, they are not obligatory, in fact, they are exceptionally rare when a number of layers is small. So the mean square merit function is widely applied in synthesis.

The mean square merit function is often entered by non-negative weight function $v(\lambda)$ :

$$
\begin{equation*}
F(X)=\int_{\lambda_{d}}^{\lambda_{u}} v(\lambda)[R(X, \lambda)-\tilde{R}(\lambda)]^{2} d \lambda \tag{2.1.2}
\end{equation*}
$$

The introduction of the weight function allows to estimate the proximity of $R(X, \lambda))$ and $\tilde{R}(\lambda)$ in different ways at various parts of the spectrum and thus achieve a closer proximity between $R(X, \lambda)$ and $\tilde{R}(\lambda)$ in most vital parts at the expense of a possible decrease of the approximation precision in others.

Many problems require homogeneous approximation to the target function $\tilde{R}(\lambda)$. For instance, when designing achromatic antireflection coatings, a possibly more even kind of the spectral dependence of the residual reflection is sought for. From the point of view of these problems the so-called even estimation would be a most adequate form of the merit function:

$$
\begin{equation*}
F(X)=\max _{\lambda_{d} \leq \lambda \leq \lambda_{u}}|R(X, \lambda)-\tilde{R}(\lambda)| \tag{2.1.3}
\end{equation*}
$$

But this merit function is exceptionally rarely applied in the optical coating synthesis. This is accounted for by a relatively complicated way of the merit function optimization. The maximum-type function is not differentiable with respect to $X$ and so most widely spread optimization methods employing the $F(X)$ derivatives are not applicable here. At the same time it should be noted that the synthesis methods based on the (2.1.3) type merit function are quite possible. Such methods are successfully elaborated for solving synthesis problems in radiophysics (Katz, Meshanov and Feldstein, 1984).

Alongside with the possibility of employing merit function (2.1.3) an alternate method of obtaining a more homogeneous estimation exists. It is sufficient to slightly change the form of the function (2.1.2) for the purpose. Let us substitute the power index 2 in (2.1.2) for a larger integer $n$ :

$$
\begin{equation*}
F(X)=\int_{\lambda_{d}}^{\lambda_{u}} v(\lambda)[R(X, \lambda)-\tilde{R}(\lambda)]^{n} d \lambda \tag{2.1.4}
\end{equation*}
$$

It can easily be seen that the merit function becomes under these conditions more sensitive to the non-uniform difference between $R(X, \lambda)$ and $\tilde{R}(\lambda)$. A strict mathematical statement is valid that when $n \rightarrow \infty$ the merit function (2.1.4) with $v(\lambda)=1$ turns into an even estimation (2.1.3). Thus, when the resulting spectral dependence is not adequately uniform in approximating the sought one it is expedient to just increase the power index in (2.1.2). This is, calculationally, a simpler way, as the function (2.1.4) is differentiable with respect to $X$.

In a numerical solution of the synthesis problem, it is impossible to calculate $R(X, \lambda)$ at all points of the spectral band. So, let us introduce some set of wavelength values for the band. Let the total number of points in the set is equal to $L$. Let us denote the points as $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{L}$. Below we will discuss the selection of the points and their location when we consider numerical examples of synthesis.

Integrals are always calculated on the computer by some approximate formula. All approximate formulas turn integral calculation to summing on the given wavelength set. In this connection, it is natural to write down the merit function from the very beginning as

$$
\begin{equation*}
F(X)=\sum_{l=1}^{L} v_{l}\left[R\left(X, \lambda_{l}\right)-\tilde{R}\left(\lambda_{l}\right)\right]^{2}, \tag{2.1.5}
\end{equation*}
$$

where $v_{l}$ are the given non-negative weight factors.
Below we will always hold that the merit function takes the form (2.1.5). The weight factors $v_{l}$ at the beginning of the synthesis axe usually assumed to be equal to a unit. Further on, their values can be adjusted depending on the results obtained (see the remark concerning the weight function selection above).

### 2.1.2 Merit function optimization

Synthesis methods based on the merit function optimization, actually bring the search for the solution down to one-time or re-iterated minimization of the function. Different versions of the general pattern of the solution search will be considered below. For the present, we will confine ourselves to the merit function optimization process proper.

A number of good books deal with the optimization methods (see, e.g., Himmelblau, 1972). So we will not dwell into mathematical details and discuss the whole diversity of the methods suitable for the synthesis problems. Let us consider just a few basic methods, and discuss primarily the calculative aspects, specific of the multilayer optical coatings synthesis problems.

In our experience various gradient methods are efficient for a search of the merit function minimum. All the methods of the type imply re-iterated calculations of the merit function and its gradient, i.e., the vector comprised by partial derivatives of the merit function with respect to the layers sought for parameters. These two operations are essential, and so the rate and precision of their fulfillment determine to a considerable degree the calculation potentials of the method.

Recurrent methods of spectral coefficients calculation considered in Chapter 1 enable a high speed and calculation precision in determining the merit function. With the normal incidence and no absorption in the layers recurrent formulas for the amplitude reflectance takes the following form ${ }^{4}$ :

$$
\begin{equation*}
r_{1,0}=\frac{n_{1}-n_{0}}{n_{1}+n_{0}}, \quad r_{j+1,0}=\frac{r_{j+1, j}+r_{j, 0} e^{-2 i \varphi_{j}}}{1+r_{j+1, j} r_{j, 0} e^{-2 i \varphi_{j}}}, \quad(j=1, \ldots, m), \tag{2.1.6}
\end{equation*}
$$

where

$$
\begin{equation*}
r_{j+1, j}=\frac{n_{j+1}-n_{j}}{n_{j+1}+n_{j}}, \quad \quad \varphi_{j}=\frac{2 \pi}{\lambda} n_{j} d_{j} . \tag{2.1.7}
\end{equation*}
$$

In these formulas, $n_{0}=n_{s}$ is the substrate refractive index, $n_{m+1}=n_{a}$ is the outer space refractive index. The layer refractive indices $n_{1}, \ldots, n_{m}$ alternately take the values of $n_{H}$ and $n_{L}$.

The reflectance is expressed through $r$ according to the formula:

$$
\begin{equation*}
R=\left|r_{m+1,0}\right|^{2}=r_{m+1,0}^{*} r_{m+1,0} \tag{2.1.8}
\end{equation*}
$$

In order to calculate the merit function (2.1.5), it is necessary to calculate $R\left(X, \lambda_{l}\right)$ reflectance values $L$ times at $\lambda=\lambda_{1}, \lambda_{2}, \ldots, \lambda_{L}$ following the formulas (2.1.6)-(2.1.8). Here, the values of the layer thicknesses in (2.1.7) are set by the $X$-vector coordinates.

Let us now discuss calculation of the merit function gradient:

$$
\nabla F(X)=\left\{\frac{\partial F}{\partial d_{1}}, \ldots, \frac{\partial F}{\partial d_{m}}\right\} .
$$

Frequently approximate formulas are applied to calculate the $F$-function derivatives:

$$
\frac{\partial F}{\partial d_{k}} \cong \frac{F\left(d_{1}, \ldots, d_{k}+\Delta d_{k}, \ldots, d_{m}\right)-F\left(d_{1}, \ldots, d_{k}, \ldots, d_{m}\right)}{\Delta d_{k}}
$$

It is evident in this case that in order to determine $\nabla F$ it is necessary to additionally calculate the merit function m times alternately giving small increments to all the thicknesses. This, naturally, results in an extended calculation time. Note also, that in this way the derivatives are sometimes calculated with a low precision level, especially near the local minima of the merit function.

So it is preferable to use another way of calculating $\nabla F$. It is based on the precise analytical expressions for the merit function partial derivatives. Let us deduce these expressions for the function (2.1.5) under investigation and selected method of the reflectance calculation.

Let the thickness of the k-th layer change by the small value $\delta$. Determine the major increment term of the amplitude reflectance with respect to $\delta$. It is evident that with the thickness variation of the $k$-th layer $r_{j+1,0}$ changes if $j \geq k$. It is easy to calculate that the $r_{k+1,0}$ change is equal to

$$
\begin{equation*}
\Delta r_{k+1,0}=-i \frac{4 \pi}{\lambda} n_{k} r_{k, 0} e^{-2 i \varphi_{k}} \frac{1-r_{k+1, k}^{2}}{\left(1+r_{k+1, k} r_{k, 0} e^{-2 i \varphi_{k}}\right)^{2}} \delta . \tag{2.1.9}
\end{equation*}
$$

With $j>k$ the increments of $r_{j+1,0}$ can be expressed through $r_{j, 0}$ increment. Then we obtain

$$
\begin{equation*}
\Delta r_{j+1,0}=e^{-2 i \varphi_{j}} \frac{1-r_{j+1, j}^{2}}{\left(1+r_{j+1, j} r_{j, 0} e^{-2 i \varphi_{k}}\right)^{2}} \Delta r_{j, 0} \tag{2.1.10}
\end{equation*}
$$

Let us introduce the variable $\Psi_{j}$ setting a recurrent formula for $\Psi_{j}$ calculation in the opposite, compared to $r_{j, 0}$, direction:

$$
\begin{equation*}
\Psi_{j}=e^{-2 i \varphi_{j}} \frac{1-r_{j+1, j}^{2}}{\left(1+r_{j+1, j} r_{j, 0} e^{-2 i \varphi_{k}}\right)^{2}} \Psi_{j+1} \tag{2.1.11}
\end{equation*}
$$

Let us call it a conjugate variable. The initial value for $\Psi_{j}$, that is the $\Psi_{m+1}$ value will be determined later. The validity of the following equation follows from (2.1.10), (2.1.11):

$$
\begin{equation*}
\Psi_{j} \Delta r_{j, 0}=\text { const } \tag{2.1.12}
\end{equation*}
$$

in other words, independence of the product from the $j$-number. Equation (2.1.12), in particular, gives

$$
\Psi_{m+1} \Delta r_{m+1,0}=\Psi_{k+1} \Delta r_{k+1,0}
$$

Substituting expression (2.1.9) for $\Delta r_{k+1,0}$ here and taking account of formula (2.1.11) with $j=k$, we obtain

$$
\begin{equation*}
\Psi_{m+1} \Delta r_{m+1,0}=-i \frac{4 \pi}{\lambda} n_{k} r_{k, 0} \Psi_{k} \delta \tag{2.1.13}
\end{equation*}
$$

With the changing thickness of the $k$-th layer the reflectance changes at all wavelengths. Let us denote $R$-increment at $\lambda=\lambda_{l}$ as $\Delta R^{(l)}$. Further, we will supply all the values calculated for $\lambda=\lambda_{l}$ with the upper $l$-index in round brackets. In particular, $r_{j+1,0}^{(l)}$ are values calculated from recurrent formulas (2.1.6) with $\lambda=\lambda_{l}$ in (2.1.7).

In accordance with (2.1.8)

$$
\begin{equation*}
\Delta R^{(l)}=2 \operatorname{Re}\left\{\left(r_{m+1,0}^{(l)}\right)^{*} \Delta r_{m+1,0}^{(l)}\right\} . \tag{2.1.14}
\end{equation*}
$$

The merit function increment caused by the variation of the $k$-th layer thickness is equal to

$$
\Delta F=2 \sum_{l=1}^{L} v_{l}\left[R\left(X, \lambda_{l}\right)-\tilde{R}\left(\lambda_{l}\right)\right] \Delta R^{(l)}
$$

or, taking account of (2.1.14),

$$
\begin{equation*}
\Delta F=\sum_{l=1}^{L} \operatorname{Re}\left\{4 v_{l}\left[R\left(X, \lambda_{l}\right)-\tilde{R}\left(\lambda_{l}\right)\right]\left(r_{m+1,0}^{(l)}\right)^{*} \Delta r_{m+1,0}^{(l)}\right\} \tag{2.1.15}
\end{equation*}
$$

Let us set the initial conditions for the recurrent formula (2.1.11) $L$-times:

$$
\begin{equation*}
\Psi_{m+1}^{(l)}=4 v_{l}\left[R\left(X, \lambda_{l}\right)-\tilde{R}\left(\lambda_{l}\right)\right]\left(r_{m+1,0}^{(l)}\right)^{*} \tag{2.1.16}
\end{equation*}
$$

and let us carry out the conjugate variables calculation for all $j$ from $m$ to $1 L$ times. The conjugate variables calculated at the $l$-th initial conditions will also be supplied with the $l$ upper index in brackets, as $\Psi_{j}^{(l)}$.

Using (2.1.16), let us put down expression (2.1.15) for the merit function increment as:

$$
\begin{equation*}
\Delta F=\sum_{l=1}^{L} \operatorname{Re}\left\{\Psi_{m+1}^{(l)} \Delta r_{m+1,0}^{(l)}\right\} \tag{2.1.17}
\end{equation*}
$$

Equality (2.1.13) is valid at any $\lambda$. Thus, when $\lambda=\lambda_{l}$ we will obtain

$$
\begin{equation*}
\Psi_{m+1}^{(l)} \Delta r_{m+1,0}^{(l)}=-i \frac{4 \pi}{\lambda_{l}} n_{k} r_{k, 0}^{(l)} \Psi_{k}^{(l)} \delta \tag{2.1.18}
\end{equation*}
$$

Substituting (2.1.18) into (2.1.17) we find that

$$
\Delta F=\sum_{l=1}^{L} \frac{4 \pi}{\lambda_{l}} n_{k} \operatorname{Im}\left\{r_{k, 0}^{(l)} \Psi_{k}^{(l)}\right\} \delta
$$

It follows from the above, that a partial derivative of the merit function with respect to the $k$-th layer thickness is equal to

$$
\begin{equation*}
\frac{\partial F}{\partial d_{k}}=\sum_{l=1}^{L} \frac{4 \pi}{\lambda_{l}} n_{k} \operatorname{Im}\left\{r_{k, 0}^{(l)} \Psi_{k}^{(l)}\right\} \tag{2.1.19}
\end{equation*}
$$

The resulting expressions allow a fast computation of the merit function gradient. Actually, in order to calculate all partial derivatives following formula (2.1.19), it is sufficient to have the values of the conjugate variable obtained as a result of $L$-operations on the recurrent
formula (2.1.11) from $m$ to 1 . These operations consume as long as a single calculation of the merit function, (since it also requires $L$ operations of formula (2.1.6) from 1 to $m$ ). Mathematically, the resulting expressions are absolutely precise. Hence, a computer precision of calculations is achieved. Due to this, the use of (2.1.11), (2.1.16) and (2.1.19) in calculations increase the efficiency of the optimization methods employed.

In our experience, a high speed and precision in merit function minimum determining are achieved when the combination of steepest descent method and the conjugate gradients method is used. Optimization from the starting design $X_{0}$ begins by employing the first one and then the second one is used as the minimum is approached. The starting design choice will be considered in detail in the next subsection. Meanwhile we will dwell at some length on the gist of the methods enumerated above.

Vector - $\nabla F$, opposite in its direction to the merit function gradient, is known as its antigradient. It shows the direction of the fastest decrease of the $F$-function at every point within its area of determination. In the steepest descent method, the descent occurs from the starting point $X^{0}$ in the direction of $-\nabla F\left(X^{0}\right)$ antigradient. It means that the minimum of a single variable function

$$
\begin{equation*}
\varphi(\alpha)=F\left[X^{0}-\alpha \nabla F\left(X^{0}\right)\right] \tag{2.1.20}
\end{equation*}
$$

is being sought for along the $X^{0}-\alpha \nabla F\left(X^{0}\right)$ straight line in the parameter space.
There are numerous ways of seeking the function $\varphi(\alpha)$ minimum value, or, as it is put, the ways of one-dimension minimization. The simplest one looks as follows. A certain step value $h$ is set along the antigradient direction. Then a sequential motion along this direction is effected in the parameter space. This motion is taken in $-h \nabla F$ steps. It means that when taking the $p$-th step we find ourselves at the $X^{0}-p h \nabla F\left(X^{0}\right)$ point. Every step is accompanied by a verification of whether the $\varphi$-function keeps decreasing, i.e., whether the condition $\varphi(p h)<\varphi[(p-1) h]$ is valid. As soon as the condition is violated, one step backward is taken and the point of the stop is taken for the minimum coordinate along the set antigradient direction.

The simplest algorithm of a one-dimension minimization can be improved. For a higher precision a parabolic interpolation of the $\varphi(a)$ function is possible using the three last points and determining the minimum point on the obtained parabola. A step-counter is of use, too. If the steps are too many, which tells adversely on the solution time it is expedient to increase the $h$ -
value of the step. If the steps are too few which tells adversely on the precision of the onedimension minimization or if the very first step results in increasing the $F$-function, it is necessary to decrease the $h$-value of the step.

Let us denote the minimum point obtained in the process of one-dimension minimization as $X^{1}$ The function $F$ gradient is calculated again at point $X^{1}$, and a descent along the newlyoriented antigradient occurs. The next minimum point obtained in the process of the onedimension minimization can be denoted as $X^{2}$. As a result of a series of similar descents, we obtain a sequence of points

$$
X^{0}, X^{1}, \ldots, X^{k}, \ldots
$$

with the merit function $F$ values monotonously decreasing along the sequence:

$$
F\left(X^{0}\right)>F\left(X^{1}\right)>\ldots>F\left(X^{k}\right)>\ldots
$$

Various criteria are used for interrupting the steepest descent method, the most widely used one taking into account a merit function change. This is the way it is implemented. A certain relatively small number $\varepsilon$ is set. As soon as the following inequality becomes valid

$$
\frac{F\left(X^{k-1}\right)-F\left(X^{k}\right)}{F\left(X^{k-1}\right)}<\varepsilon,
$$

the method terminates.
Usually, the steepest descent method predetermines a fast decrease of the function along the first few directions of one-dimension minimizations. Then the speed of the decrease slows radically. This is especially evident when the function isolines have a "ravine-like" structure. Experience shows that the merit function isolines in the synthesis problems look exactly like the former. The "ravine-like" pattern of the merit function grows fast as the number of the $F$ function variables, i.e., the number of the layers increases. The conjugate gradients method allows to manage the ravine-like pattern of the minimized function in a simple and reliable way.

The conjugate gradients method differs from the steepest descent one in the fact that a one-dimension minimization is not implemented along the antigradient but rather along some "'adjusted" direction obtained with the previous descent direction taken into account. Let $X^{k}$ be a point obtained after the $k$-th process of a one-dimension minimization; $\nabla F\left(X^{k}\right)$ be the merit function gradient value at this point. Let us use designation of $p_{k+1}$ for the vector pointing the
next $k+1$-st direction of minimization. The antigradient vector at the starting point $X^{0}$ is selected as the first direction:

$$
p_{1}=-\nabla F\left(X^{0}\right) .
$$

The subsequent directions of minimization are determined in the following way:

$$
p_{k+1}=-\nabla F\left(X^{k}\right)+\beta_{k} p_{k}
$$

where the $\beta_{k}$ factor, adjusting the antigradient direction towards the previous direction of descent is determined from the formula

$$
\beta_{k}=\frac{\left(\nabla F\left(X^{k}\right), \nabla F\left(X^{k}\right)-\nabla F\left(X^{k-1}\right)\right)}{\left|\nabla F\left(X^{k}\right)\right|^{2}} .
$$

The outer round brackets denote here the scalar product of vectors.
The process of a one-dimension minimization in the conjugate gradients method can be carried out in different ways. Thus, the above method can be applied, too.

It is sensible to have the so-called renewal of the conjugate gradients method from time to time. It means, in effect, that after every $s$ descents, the $\beta_{s+1}, \beta_{2 s+1}, \ldots$ factors are believed to be equal to zero, and the antigradient vector is selected as the next minimization direction. The s number is usually determined within the range of 5-10. The method renewing decreases the impact of calculation errors, including errors of a one-dimension minimization.

As a criterion of a transition to the conjugate gradients method, a relative decrease of a merit function in the successive one-dimension minimizations can be used. In our experience, the transition is reasonable when the decrease becomes less than $5 \cdot 10^{-2}-5 \cdot 10^{-3}$.

Let us consider another important item. To simplify the problems, we have so far never mentioned any limitations concerning permissible values of the coating parameters. However, they do exist, the major of them being positive values of the thicknesses of the coating layers. We will consider it in more detail. There may be other limitations too, one of them, e.g., dealing with the total thickness of the coating, but they are often non-essential.

Numerous optimization methods taking account of the limitations in the parameter space have been elaborated. For their detailed description one can see D. Himmelblau's book (1972). Here we will describe a simplest way of taking account of the positive values of the layer thicknesses.

Let $p$-vector be setting the direction of a one-dimension minimization in the method currently employed. Let us put it down in the coordinate form as

$$
p=\left\{p_{1}, \ldots, p_{m}\right\}
$$

In the process of one-dimension minimization the motion towards the minimum value is effected as successive steps of the following type:

$$
X^{(v+1)}=X^{(\nu)}-h p,
$$

where $h$ is the value of the method step, $X^{(\nu)}$ and $X^{(\nu+1)}$ are the coating parameters vectors at the beginning and the end of every successive step. The coordinate form of the expression is:

$$
d_{k}^{(v+1)}=d_{k}^{(v)}-h p_{k}, \quad(k=1, \ldots, m)
$$

If, as a result of any step, some layer thickness $d_{k}^{(\nu+1)}$ becomes negative, its value is substituted by zero. Otherwise, the one-dimension minimization remains similar to the description above. As a result, condition $d_{k} \geq 0$ for all $k=1, \ldots, m$ is always observed. The described way of taking account of the permissible parameters values is actually a simplest version of the gradient projection method.

### 2.1.3 General pattern of the synthesis and the problem of the starting design choice

As we set the problem, it is natural to consider optimal for the given number of layers a solution which corresponds to the global minimum of the merit function.

However, the merit function is multiextremal, and besides the global minimum there exist a lot of local minima. Their number grows fast as the number of layers grows and reaches hundreds and thousands even for 6-10 layer coatings. This makes a search for an optimal solution extremely complicated.

Note also, that the very concept of the optimal solution is but poorly formalized. Indeed, which is better: a multilayer coating possessing reflectance close to the required function $\tilde{R}(\lambda)$ or a coating with fewer layers but possessing reflectance somewhat less approximating the above function? It is even more ambiguous, as it is next to impossible to guess in advance the relation between the number of layers and the approximation closeness to the required function.

In solving any kind of the synthesis problems, it is necessary to take into account a number of other non-quantitative aspects, including, firstly, the necessary costs of the solution in terms of computer time and labors costs. A waste of many hours in search of an optimal solution
is evidently senseless if, on the other hand, we can find within several minutes "an acceptably adequate" solution satisfying all the requirements.

All the above causes the following conclusion concerning the merit function optimization: a global optimization is frequently redundant and it is advisable to look for a sufficiently thorough local minimum ensuring the required accuracy of the pre-set spectral dependence approximation and good feasibility properties of the coating design.

Such "quasioptimal" solution is frequently achieved due to a good selection of a starting design for the merit function optimization. The researcher's personal experience is undoubtedly of utmost importance, though a number of common rules can be recommended. This, firstly, concerns, edge filters of various type. It is advisable to use quarter-wave mirrors or some combinations of such mirrors as the first approximation to their design.

The required spectral characteristics of the edge filters have two specific adjacent areas: that of a high reflectance and the one of a high transmittance. In synthesis, $\tilde{R}(\lambda)$ is usually set as equal to 0 in one single spectral band and $\tilde{R}(\lambda)=1$ in the other one, adjacent to the former. We showed in Chapter 1 that the layer thickness variations are of little influence on the spectral characteristics of the multilayer systems in the high reflectance zones, while they are much more pronounced in the adjacent areas. So, by varying the layer thicknesses, we can increase transmittance beyond the high reflectance zones of the mirrors without any noticeable decrease of their reflection properties. It is due to this property that quarter-wave mirrors and their combinations as the starting designs for the given class of synthesis problems are preferable.

The properties of the mirrors have been thoroughly studied (see Chapter 1). Thus, it is easy to select and match their parameters so that a sufficient reflectance level can be ensured within the required band (the parameters include the number of layers, and the central wavelength). If the band turns to be of a wider range than the principal high-reflectance zone of the mirror, it is necessary to employ superimposed quarter-wave mirrors with central wavelengths displaced with respect to each other as the starting design.

As an example, Fig. 2.1 shows the reflectance of an edge filter obtained in the described way. The layer refractive indices are $n_{H}=2.30, n_{L}=1.34$, the substrate refractive index is 1.52 , the outer space is the air $\left(n_{a}=1\right)$. The layer adjacent to the substrate has a high refractive index. The filter was designed for the use as a cold mirror in light sources.


Figure. 2.1: Reflectance of the 21-layer edge filter
21S . 798 H 1.217 L 1.310 H 1.351 L 1.185 H 1.451 L 1.389 H 1.207 L 1.513H 1.246L
1.062H 1.174L . 674H . 785 L . 999 H . 940 L 1.142H . 873 L . 521 H 1.490 L .289 H
with $n_{s}=1.52, n_{H}=2.30, n_{L}=1.34\left(\lambda_{0}=500 \mathrm{~nm}\right)$

In synthesis, $\tilde{R}(\lambda) \equiv 1$ was given within the band between 400 and 800 nm and $\tilde{R}(\lambda) \equiv 0$ within the band between 800 and 2000 nm . Since the high-reflectance zone of a quarter-wave mirror with refractive indices equal to 2.30 and 1.38 does not overlay the required spectral band, a combination of three 7-layer mirrors featuring central wavelengths of 700, 600 and 500 nm was used. It is of interest to note here that as a result of optimization, the merit function value was decreased a hundred-fold.

Using this example, let us consider some ideas concerning the number of layers of a synthesized coating, the wavelength set within the spectral band in question and weight factors in (2.1.5).

The number of layers of the mirrors presenting the starting design was determined so that an adequate level of reflectance was ensured within the required spectral band satisfying the elaborators of the light sources.

As for the selection of the wavelength set, two conflicting requirements exist. On the one hand, there is a wish to describe the obtained spectral dependencies with utmost precision, and hence there is a necessity to use dense sets. On the other hand, it is not desirable to extend the calculation time, which is proportional to the number of points in the set. So we have to employ
compromise options. In order to achieve an adequate degree of accuracy of the spectral curve description, it is usually sufficient to take 3-4 points of the set between its two adjacent extreme points. We should keep in mind that the distance between two adjacent extreme points decreases as the number of layers grows. An experience of selecting sets is usually accumulated fast in practical solution of synthesis problems. For this purpose, preliminary calculations on dense sets can be used. In many cases it is worthwhile to use non-homogeneous sets in order to save the calculation time. They can frequently be selected as less thick in the long-wave part of the spectral band as the oscillation swing of the spectral curve decreases here. For instance, in order to calculate the design in Fig. 2.1 a set of a 20 nm steps was used in the area between 400 and 800 nm , and a 40 nm step was employed within the area between 800 and 2000 nm .

When non-homogeneous sets are employed, we should keep in mind that a decrease in the number of points within any spectral band leads to decreasing its relative contribution into the merit function, and, consequently, the response of the latter to the approximation accuracy of the required spectral dependence also decreases here. As was mentioned earlier, weight factors can be used to regulate the degree of closeness between the actual and the required spectral characteristics in various spectral zones. It is worthwhile to change them sometimes as synthesis goes, if it becomes evident in the process of optimization that the approximation accuracy at some essential spectral zones is not sufficient. When such systems as edge filters are synthesized, it is worthwhile to select weight factors small or even equal to zero, in the areas of sharp changes of the target spectral curve. Otherwise, these areas can domineer as a contribution into the merit function due to the great discrepancy between the actual and the required characteristics. In our example, the vicinity of the $\lambda$ point equal to 800 nm represents such an area.

It is difficult to give all-embracing recommendations concerning the starting design choice and algorithm peculiarities for different types of synthesis problems. Let us only note here that good starting designs can be also set in case of wide-band mirrors and polarizers synthesis.

Let us now discuss synthesis problems where the starting design choice is vague, as a rule. Such are the problems of synthesis of antireflection coatings, neutral beam splitters, multilayer systems featuring non-standard spectral properties. How can a "quasioptimal" solution, if not the optimal one be obtained for them?

Let us make another remark concerning optimization of multiextremal functions. In principle, there are mathematical methods of their global minimum search. However, a strict employment of these methods which would ensure determining the global minimum is not
feasible practically when the number of variables exceeds 1 or 2 because it implies the necessity of tremendous amounts of computer time. Due to this fact, methods are mostly used which, speaking strictly mathematically, cannot guarantee unconditional determining of the global minimum, but they are completely justifiable in solving practical problems. The considerations above, related to a search of quasioptimal solution of the synthesis problems accord with this approach.

Very often multilayer optics synthesis problems are solved successfully through reiterated optimization of the merit function with series of random starting designs. As a quasioptimal solution, a vector of the coating parameters is taken, which corresponds to the deepest of the obtained minima. Sometimes, it is worthwhile to pick out several quasioptimal solutions, especially, if the minima are about the same depths. The final selection of the best solution can be done on the basis of the results of their practical implementation.

Let us consider the calculating process in more detail. In order to set the starting designs, a generator of evenly distributed numbers is used. This code can be found on any computer both mainframes and PCs. For the sake of certainty, evenly distributed over the segment $[0,1]$ numbers are generated. Let us denote them as $\xi_{1}, \xi_{1}, \ldots, \xi_{n}, \ldots$. The coordinates of the first starting design vector are set in the following way:

$$
d_{1}=D \xi_{1}, \ldots, d_{m}=D \xi_{m}
$$

where $D$ is a normalization factor whose choice will be discussed a little later. The second starting design employs the following random numbers $\xi_{m+1}, \ldots, \xi_{2 m}$, etc.

Certain conclusions can be made concerning a selection of the normalization factor if we consider the geometric interpretation of the admittance supplied in Section 1.2. It was shown there that with a change of the layer thickness from 0 to $\lambda /(2 n)$, the admittance makes a complete circle in its phase plane. With a further growth of the layer thickness, the same values are obtained. Thus, if we meant synthesis at one wavelength, employing a thicker layer than $\lambda /(2 n)$ would be just senseless. But this cannot be so categorically asserted when a synthesis in the spectral band is involved since we have different trajectories in the admittance phase plane for different wavelengths. Nevertheless, practical experience shows that in the latter case, it is sensible to take similar limitations concerning layer-thicknesses of the starting design. Hence, we obtain the following estimation for the normalization factor selection:

$$
D \sim \frac{\lambda}{2 n} .
$$

Here $\lambda$ can be substituted for the mean wavelength of the spectral band in question, and $n$ - for the mean value of the layer refractive indices.

The number of the starting designs under consideration can be limited by setting either their total number or the calculation time. In order to try to obtain deepest local minimum, it is worthwhile to do it in two steps. First a rough (and consequently a much faster, rather than more accurate) optimization of the merit function is effected for a series of randomly set starting designs. The stop criterion for it can be determined as a relative decrease of the function by $5 \cdot 10^{-3}-10^{-2}$ value. Further a more accurate optimization follows for only 5-10 options where the first stage resulted in least values of the merit function.

So far we have spoken about a search of the merit function minimum at a set number of the coating layers. Let us now consider the question of setting this number. Only most general recommendations are possible here, as the selection of the number of layers is determined, firstly, by concrete design requirements and, secondly, it depends greatly on the course of the solution (i.e., on where it is possible or impossible to reach a desired level of approximation of the target spectral dependence). We imply problems where assessment of the number of layers necessary to achieve the pre-set spectral properties is vague or not clear at the beginning.

On the whole, it is worthwhile to look for the solution of the problem by gradually increasing the number of layers as the difficulties of optimization increase considerably with their number growing. However, it is reasonable sometimes to consider a relatively large m from the very beginning, as it enables a simultaneous search for designs with a various number of layers. Let us dwell on it in more detail. For instance, when designing antireflection coatings, we tend to obtain a series of antireflection coatings with various numbers of layers, still not exceeding, say, six. Then it is worthwhile to use as many starting designs with six layers as possible at the first, rough stage. Since some layer thicknesses generated by the random number generator, are small, many of them turn to 0 in the process of optimization (due to the projection to the permissible parameters region). If the thickness of the outer layer turns to zero, the total number of layers decreases by one. If the thickness of some middle layer turns to 0 , the adjacent layers having the same refractive indices, merge, thus decreasing the total number of layers by 2. Due to this fact, the very first, rough, stage of the optimization gives a number of systems with
$m<6$. They can be further used as starting designs for synthesis of antireflection coatings with a number of layers fewer than 6 .

In conclusion of the section, let us consider the result of the synthesis of six-layer antireflection coatings for the spectral band between 400 and 900 nm . Materials featuring $n_{H}=2.30, n_{L}=1.45$ were used in the synthesis. The layer adjacent to the substrate has a high refractive index. The substrate has $n_{s}=1.52$, the outer space is the air. The normalization factor for generating layer thicknesses in compliance with the above consideration was taken equal to 150 nm . During the first, rough stage of optimization with the stop criterion of the function relative decrease equal to $5 \cdot 10^{-3}$, over 100 starting designs were generated. A further, more refined optimization with a stop at the $10^{-4}$ value of the relative function decrease was carried out for 10 options with the minimal values of the merit function obtained in the coarser optimization stage. Two final (quasioptimal) solutions were selected for the least values of the merit function. The reflectances of the obtained solutions are presented in Fig. 2.2.


Fig. 2.2: Reflectances of the 6-layer antireflection coatings with (1) 6S . 318 H .34 L 1.977 H .106 L .375 H 1.099 L , (2) 6S . 34 H .391 L 1.104 H .188 L .710 H 1.216 L with $n_{s}=1.52, n_{H}=2.30, n_{L}=1.45\left(\lambda_{0}=500 \mathrm{~nm}\right)$

### 2.2 Synthesis method based on the needle-like variations of the refractive index

The synthesis method considered at the end of the previous section enables to find a reliably good solution within a sensible time span in case the number of layers is small and
limited to 6-8. As the number of layers grows, the number of local minima grows dramatically and a probability to reach a sufficiently deep minimum decreases significantly. With the number of layers growing, the rate of search for the local minimum slows down and becomes more complicated due to a more complex structure of the function and an extended time of the function and the gradient calculations. Sometimes these difficulties are overcome by varying thicknesses of only a few layers rather than all of them in the course of synthesis. This can be done, for instance, when there is a high reflectance zones in the required spectral band. Then, it is worthwhile to include fixed quarter-wave mirrors providing high reflection in the required zones into the designs in order to change the thicknesses of only some additional layers. However, on the whole, this approach cannot guarantee common grounds and by far not always produces satisfactory results. The method considered below, according to a vast experience of its practical application, makes it possible to find a good solution for many cases when other methods fail to.

The idea of the method was first conceived when conditions for optimal solutions of synthesis problems were investigated (Tikhonravov, 1982). This method was later successfully employed in solving various synthesis problems at normal and oblique light incidence, including special cases when it was necessary to obtain simultaneously the pre-set energy and phase properties of the coating. We will consider here the simplest version of the method designed for a synthesis of two-component multilayer coatings at the normal light incidence. Let us first discuss principal ideas necessary to create it. At the end of the section we will try to supply some mathematical explanations to the success of the employment of the method.

### 2.1.1 Needle-like variations of the refractive index

In order to introduce the principal ideas of the method, it is convenient to describe a multilayer coating by some distribution of the permittivity $\varepsilon(z)$ (see Section 1.1 ). We will consider that there is no absorption in the layers, so in equations of Section $1.1 \varepsilon(z)=n^{2}(z)$, where $n(z)$ is a function describing a distribution of refractive index. Two-component coatings have piecewise continuous functions $n(z)$ alternatively taking the values of $n_{H}$ and $n_{L}$. So far it is convenient to consider $n(z)$ as an arbitrary piecewise continuous function. In order to determine the spectral characteristics of the coating, the admittance method can be used. With
the normal light incidence, the amplitude reflectance is expressed through the input admittance according to the formula

$$
\begin{equation*}
r(\lambda)=\frac{n_{a}-A\left(z_{a}, \lambda\right)}{n_{a}+A\left(z_{a}, \lambda\right)}, \tag{2.2.1}
\end{equation*}
$$

where for coatings with arbitrary $n(z)$ distribution admittance is obtained as a solution of the differential equation

$$
\begin{equation*}
\frac{d A}{d z}=i \frac{2 \pi}{\lambda}\left[n^{2}(z)-A^{2}(z, \lambda)\right] \tag{2.2.2}
\end{equation*}
$$

with the initial condition

$$
\begin{equation*}
A(0, \lambda)=n_{s} . \tag{2.2.3}
\end{equation*}
$$

Remember that 0 is a coordinate of the boundary between the coating and the substrate, $z_{a}$ is a coordinate of the boundary with the outer space. Thus, $z_{a}$ is a total geometrical thickness of the coating.

The reflectance is expressed according to

$$
\begin{equation*}
R(\lambda)=|r(\lambda)|^{2}=r(\lambda) r^{*}(\lambda) . \tag{2.2.4}
\end{equation*}
$$

It is evident from (2.2.1)-(2.2.4) that the values of the amplitude and the energy reflection coefficients are fully defined by the $n(z)$ function, which, consequently, plays a role similar to that of the $X$ vector in the previous section. So, like in Section 2.1, it will be sensible to introduce designation $R[n(z), \lambda]$ for energy reflection coefficient. Let us also introduce the merit function similar to (2.1.5):

$$
\begin{equation*}
F[n(z)]=\sum_{l=1}^{L} v_{l}\left\{R\left[n(z), \lambda_{l}\right]-\tilde{R}\left(\lambda_{l}\right)\right\}^{2} . \tag{2.2.5}
\end{equation*}
$$

Its value is non-ambiguously determined by the $n(z)$ function.
When solving synthesis problems, we actually match the $n(z)$ function, so that the merit function (2.2.5) can be minimized. It certainly means that only some functions of the so-called permissible set are matched. For instance, when designing two-component coatings only piecewise continuous functions are considered taking alternately the values of $n_{H}$ and $n_{L}$.

This view of the synthesis problems make them similar to the classical problems of the optimal control theory (Pontrjagin et al., 1976). The merit function (2.2.5) turns to be goal function of the optimal control theory, the $n(z)$ refractive index distribution is similar to the
control function and the differential equation for admittance (2.2.2) is a sort of motion equation guided by $n(z)$. Let us note, by the way, that the fruitfulness of the latter comparison was demonstrated in Section 1.2. At the same time, there are certain differences: equation (2.2.2) includes the spectral parameter $\lambda$, and the merit function is determined as the sum depending on the solutions of equation (2.2.2) for the various values of this parameter.

Of special importance for us is a specific construction of the optimal control theory - the so-called needle-like variation of the control function. It is illustrated in Fig. 2.3. Let $n(z)$ be some piecewise continuous distribution of the refractive index (a solid curve in the upper part of Fig. 2.3). Let us select some point $\hat{z}$ on $\left[0, z_{a}\right]$ where function $n(z)$ is continuous, and consider the small segment $[\hat{z}, \hat{z}+\Delta z]$ on its right. Let us change the refractive index $n(z)$ here, taking it equal to some value $\hat{n}$. This change of the refractive index is called a needle-like variation of the $n(z)$ function at point $\hat{z}$. The origin of the name is evident from Fig 2.3. The $n(z)$ variation takes the shape of a "needle" (hatched in Fig. 2.3): its width $\Delta z$ is small, according to the condition, while its height can be significant.

Let us find out how the solution of the equation for the admittance changes when the needle-like variation of the refractive index $n(z)$ at point $\hat{z}$ is done. Not to overcomplicate the explanation, let us not be overduly strict from the mathematical point of view. A strict substantiation of the following is similar to the proof of corresponding results in the optimal control theory.


Figure 2.3: Needle-like variation of the refractive index and corresponding admittance variation

Let us designate as $\tilde{n}(z)$ the function obtained as a result of the needle-like variation at point $\hat{z}$ :

$$
\tilde{n}(z)= \begin{cases}n(z), & z \notin[\hat{z}, \hat{z}+\Delta z], \\ \hat{n}, & z \in[\hat{z}, \hat{z}+\Delta z] .\end{cases}
$$

Let $\tilde{A}(z, \lambda)$ be a relevant solution of the equation for admittance:

$$
\begin{equation*}
\frac{d \tilde{A}}{d z}=i \frac{2 \pi}{\lambda}\left[\tilde{n}^{2}(z)-\tilde{A}(z, \lambda)\right] \tag{2.2.6}
\end{equation*}
$$

satisfying the same initial condition as before:

$$
\tilde{A}(0, \lambda)=n_{s} .
$$

Functions $n(z)$ and $\tilde{n}(z)$ are the same within the segment $[0, \hat{z}]$, so solutions of equations (2.2.2) and (2.2.6) coincide. Thus, $\tilde{A}(\hat{z}, \lambda)=A(\hat{z}, \lambda)$. Starting at point $\hat{z}$ these solutions diverge (see Fig. 2.3) ${ }^{5}$. Let us consider them first at the narrow segment $[\hat{z}, \hat{z}+\Delta z]$. The increments of $A(z, \lambda)$ nd $\tilde{A}(z, \lambda)$ function within the segment will be determined to an accuracy of the first order with respect to $\Delta z$. In this approximation the rates of changes of the solutions can be taken as constants and equal to their values at the left point of the segment. Keeping that in mind, we obtain from (2.2.2) and (2.2.6) that the solution increments within the segment $\left[\hat{z}, \hat{z}+z_{a}\right]$ are equal to

$$
\begin{align*}
& \Delta A=i \frac{2 \pi}{\lambda}\left[n^{2}(\hat{z})-A^{2}(\hat{z}, \lambda)\right] \Delta z  \tag{2.2.7}\\
& \Delta \tilde{A}=i \frac{2 \pi}{\lambda}\left[\hat{n}^{2}-A^{2}(\hat{z}, \lambda)\right] \Delta z .
\end{align*}
$$

Here, we also took into account that $\tilde{A}(\hat{z}, \lambda)=A(\hat{z}, \lambda)$, and $\tilde{n}(z)$ are equal to $\hat{n}$ at the segment in question. Subtracting the first from the second equation in (2.2.7) we obtain that solutions of equations (2.2.2) and (2.2.6) will digress at $\hat{z}+\Delta z$ by the value of

$$
\begin{equation*}
\delta A=i \frac{2 \pi}{\lambda}\left[\hat{n}^{2}-n^{2}(\hat{z})\right] \Delta z . \tag{2.2.8}
\end{equation*}
$$

Within the next segment $\left[\hat{z}+\Delta z, z_{a}\right]$ the $\tilde{n}(z)$ and $n(z)$ functions will coincide again, however the solutions for (2.2.2) and (2.2.6) will be different as they have different values at the initial point of segment $\hat{z}+\Delta z$. Let us designate

$$
\delta A(z, \lambda)=\tilde{A}(z, \lambda)-A(z, \lambda)
$$

Function $\delta A(z, \lambda)$ is small, since the difference (2.2.8) between the initial conditions at point $\hat{z}+\Delta z$ is small. It comes from the theorem on continuous dependence of the differential equation solutions on the initial conditions.

We will obtain a differential equation for the $\delta A(z, \lambda)$ function, its smallness taken into consideration. For this, let us assume $\tilde{A}(z, \lambda)=A(z, \lambda)+\delta A(z, \lambda)$ in (2.2.6) and subtract equation (2.2.2) from (2.2.6). Taking into account that within the segment in question $\tilde{n}(z)$ and $n(z)$ functions coincide, and having neglected the square of $\delta A(z, \lambda)$ we will obtain

$$
\begin{equation*}
\frac{d(\delta A)}{d z}=-i \frac{4 \pi}{\lambda} A(z, \lambda) \delta A(z, \lambda) \tag{2.2.9}
\end{equation*}
$$

It is a linear differential equation with respect to $\delta A(z, \lambda)$ function. Its initial condition is set in equation (2.2.8) at point $\hat{z}+\Delta z$. Since the smallness order of the initial condition is $\Delta z$, the whole solution of equation (2.2.9) has the same order of smallness. Further, it is convenient to hold that the initial condition was set at point $\hat{z}$ rather than at point $\hat{z}+\Delta z$. In this case, the error introduced into the solution is proportional to $\Delta z^{2}$ and is not essential for further conclusions since from the very start all calculations are made with an accuracy to the first order of $\Delta z$. So, the initial condition for equation (2.2.9) is put down as

$$
\begin{equation*}
\delta A(\hat{z}, \lambda)=i \frac{2 \pi}{\lambda}\left[\hat{n}^{2}-n^{2}(\hat{z})\right] \Delta z . \tag{2.2.10}
\end{equation*}
$$

Equation (2.2.9) with the initial condition (2.2.10) makes it possible to obtain a change in admittance when the needle-like variation of the refractive index is made at point $\hat{z}$. This solution can be written down with an explicit formula containing function $A(z, \lambda)$ under the symbol of integral. However, we will have no need of this form of presentation. So far, the following is of importance: a change of admittance can be, in principle, found at any point up to the outer boundary $z=z_{a}$.

### 2.2.2 The principal idea of the method

With the needle-like variation of the refractive index the input admittance changes and the reflectance and the merit function changes, too.

As was mentioned earlier, equation (2.2.9) with the initial condition (2.2.10) enables in principle to find the admittance change at point $z_{a}$. So let us express the merit function (2.2.5) variation with the help of $\delta A\left(z_{a}, \lambda\right)$. For this purpose, let us first write out the variations of the amplitude and the energy reflection coefficients. From (2.2.1) we obtain that with the change of the input admittance by the value of $\delta A\left(z_{a}, \lambda\right)$, the amplitude reflection coefficient gets an increment of

$$
\begin{equation*}
\delta r(\lambda)=-\frac{[1+r(\lambda)]^{2}}{2 n_{a}} \delta A\left(z_{a}, \lambda\right) . \tag{2.2.11}
\end{equation*}
$$

It follows from (2.2.4) and (2.2.11), that

$$
\begin{equation*}
\delta R(\lambda)=2 \operatorname{Re}\left\{r^{*}(\lambda) \delta r(\lambda)\right\}=-\frac{1}{n_{a}} \operatorname{Re}\left\{r^{*}(\lambda)[1+r(\lambda)]^{2} \delta A\left(z_{a}, \lambda\right)\right\} . \tag{2.2.12}
\end{equation*}
$$

The merit function change with (2.2.12) taken account of is equal to

$$
\begin{equation*}
\delta F=2 \sum_{l=1}^{L} v_{l}\left[R\left(\lambda_{l}\right)-\tilde{R}\left(\lambda_{l}\right)\right] \delta R\left(\lambda_{l}\right)=-\frac{2}{n_{a}} \sum_{l=1}^{L} v_{l}\left[R\left(\lambda_{l}\right)-\tilde{R}\left(\lambda_{l}\right)\right] \times \operatorname{Re}\left\{r^{*}(\lambda)[1+r(\lambda)]^{2} \delta A\left(z_{a}, \lambda\right)\right\} . \tag{2.2.13}
\end{equation*}
$$

To shorten the record, we do not specify here or later an evident dependence on $n(z)$. It is quite clear from the above what kind of values depend on the distribution of the refractive index.

Let us transform (2.2.13) into a more convenient expression for further use. Let us introduce the so-called conjugate equation for the purpose:

$$
\begin{equation*}
\frac{d \Psi}{d z}=i \frac{4 \pi}{\lambda} A(z, \lambda) \Psi(z, \lambda) . \tag{2.2.14}
\end{equation*}
$$

This is a linear differential equation with respect to function $\Psi$. It is complex like (2.2.2), (2.2.6) and (2.2.9). The solution of equation (2.2.2) for admittance is its coefficient. The right part of equation (2.2.14) also contains the spectral parameter $\lambda$. Similar to the admittance, we designate here the dependence of the $\Psi$ function on this parameter. We will refer to the $\Psi(z, \lambda))$ function as a conjugate one.

Let us set $L$ times the boundary condition at point $z_{a}$ for equation (2.2.14):

$$
\begin{equation*}
\Psi\left(z_{a}, \lambda_{l}\right)=\frac{2}{n_{a}} v_{l}\left[R\left(\lambda_{l}\right)-\tilde{R}\left(\lambda_{l}\right)\right] r^{*}\left(\lambda_{l}\right)\left[1+r\left(\lambda_{l}\right)\right]^{2} . \tag{2.2.15}
\end{equation*}
$$

Now the expression (2.2.13) can be rewritten with the help of (2.2.15) as

$$
\begin{equation*}
\delta F=-\sum_{l=1}^{L} \operatorname{Re}\left\{\Psi\left(z_{a}, \lambda_{l}\right) \delta A\left(z_{a}, \lambda_{l}\right)\right\} . \tag{2.2.16}
\end{equation*}
$$

Let us consider the product of $\Psi\left(z_{a}, \lambda_{l}\right) \delta A\left(z_{a}, \lambda_{l}\right)$ at arbitrary $z$ from segment $\left[\hat{z}, z_{a}\right]$. Differentiating it with respect to $z$ and taking account of (2.2.9) and (2.2.14), we obtain that

$$
\frac{d}{d z}[\Psi(z, \lambda) \delta A(z, \lambda)]=0 .
$$

Thus, the product does not depend on $z$. So

$$
\Psi\left(z_{a}, \lambda_{l}\right) \delta A\left(z_{a}, \lambda_{l}\right)=\Psi\left(\hat{z}, \lambda_{l}\right) \delta A\left(\hat{z}, \lambda_{l}\right)
$$

Substituting $\hat{z}$ for $z_{a}$ in (2.2.16) and taking into account the initial condition (2.2.10), we finally obtain that,

$$
\begin{equation*}
\delta F=2 \pi \sum_{l=1}^{L} \frac{1}{\lambda_{l}} \operatorname{Im}\left\{\Psi\left(\hat{z}, \lambda_{l}\right)\right\}\left[\hat{n}^{2}-n^{2}(\hat{z})\right] \Delta z . \tag{2.2.17}
\end{equation*}
$$

Let us introduce another designation

$$
\begin{equation*}
P(z)=2 \pi \sum_{l=1}^{L} \frac{1}{\lambda_{l}} \operatorname{Im}\left\{\Psi\left(z, \lambda_{l}\right)\right\} . \tag{2.2.18}
\end{equation*}
$$

Then (2.2.17) can be put down as

$$
\begin{equation*}
\delta F=P(\hat{z})\left[\hat{n}^{2}-n^{2}(\hat{z})\right] \Delta z . \tag{2.2.19}
\end{equation*}
$$

This is the final expression for the value of the merit function change sought for. It underlies the method of synthesis in question. Let us make another important remark before we, at last, proceed with the explanation of the method itself. The $P(z)$ function introduced in (2.2.18) is in no way dependent on the needle-like variation of the refractive index. It is only determined by the $n(z)$ function. Indeed, in order to calculate it for the $n(z)$ arbitrary dependence, it is necessary first to provide $L$ times a numerical solution of equation (2.2.2) for different values of $\lambda$ and then to integrate numerically $L$ times in the reverse direction the conjugate equation (2.2.14) with the initial conditions (2.2.15). Solutions of the equation (2.2.2) and equation (2.2.14) depend only on $n(z)$.


Figure 2.4: Needle-like variation causing the decrease of the merit function

The idea of the method will be discussed as applicable to the synthesis of two-component coatings. By this we will further consider $n(z)$ functions taking two values of $n_{H}$ and $n_{L}$. One of these functions, corresponding to a three-layer coating, is presented in Fig. 2.4. As we have just remarked, $n(z)$ defines unambiguously the $P(z)$ function, set by equation (2.2.18). We will show below that the $P(z)$ function is determined in a very simple way with the help of recurrent formulas. Suppose now, the function has already been calculated and is presented in Fig. 2.4. We will consider the needle-like variations of $\Delta n=n_{H}-n_{L}$ high. And we will hold that at points where $n(z)=n_{L}$ they may be effected only upwards, and at points where $n(z)=n_{H}$ only downwards. Let us call such needle-like variations permissible, since when they occur, the coating corresponding to the new $n(z)$ distribution remains two-component. Let us note that a permissible variation made at the boundary of the layers is equivalent to an increase in the thickness of one layer and a decrease in the thickness of the other one. If a permissible variation occurs at some interior point of a layer, it is equivalent to a change of this layer into three new ones, with the middle layer having the refractive index different from the initial one (See Fig. 2.4).

Expression (2.2.19) makes it possible to determine the merit function change at any point $\hat{z} \in\left[0, z_{a}\right]$. In expression (2.2.19) $\hat{n}=n_{H}$, if $n(\hat{z})=n_{L}$ and $\hat{n}=n_{L}$, if $n(\hat{z})=n_{H}$ by the definition
of permissible variation. At sufficiently small $\Delta z$ this variation will necessitate the merit function decrease, if $n(\hat{z})=n_{L}$, and $P(\hat{z})<0$ at the same time, or if simultaneously $n(\hat{z})=n_{H}$ and $P(\hat{z})>0$. We mention a sufficiently small width here because the accuracy of (2.2.19) increases as $\Delta z$ decreases. With a large width of a needle-like variation, higher order terms of (2.2.19), not taken into account, reveal themselves more visually and the final expression may turn wrong. As we see, knowledge of the $P(z)$ function enables implementing the merit function minimization by carrying out permissible needle-like variation of the refractive index. It is evident that the variations are most efficient at points where modulus of the $P(z)$ function is maximal. This is the principal idea of the method. We will consider one of the versions of the method in greater detail below. Now, to ensure a more profound assessment of the considerations above, let us investigate its relation to the gradient methods of the merit function optimization.

In the process of optimization by gradient methods a successive change of the layer thicknesses by small steps occurs. Such changes are evidently equivalent to a series of needlelike variations of the refractive index at the boundaries of the layers. As we said earlier, the reverse relation is valid, too. So, the decrease of the merit function due to needle-like variations at the layer boundaries is to a certain degree similar to optimization by the gradient methods. It leads to an idea that the information used in these methods is somehow present in the $P(z)$ function. It really turns out to be so. We may prove that the merit function derivatives with respect to the layer thicknesses are expressed through the values of $P(z)$ at the boundaries of the layers.

But the $P(z)$ function contains more information than the one used by the gradient methods. Let us explain it in more details. Let a coating be found in the process of the merit function optimization. Then the $P(z)$ function values at the boundaries of the layers must be close to zero. Otherwise, due to small changes of the layer thicknesses through corresponding needle-like variations, the merit function can be significantly decreased. But this runs counter to the fact that the obtained design causes the minimum. Small differences of $P(z)$ from the zero at the layer boundaries are certainly possible because minima are always found with a definite degree of accuracy. However, the proximity of the $P(z)$ to zero near the layer boundaries does not at all mean that the function is small everywhere along $\left[0, z_{a}\right]$. There may be zones inside the layers where modulus of the $P(z)$ function is sufficiently big, and permissible needlelike
variations cause significant decrease of the merit function (see above). This situation is illustrated in Fig. 2.4.

In each case of the inner needle-like variation the number of layers increases by two. If a two-component coating is mathematically described by a vector of the layer thicknesses (as in Section 2.1), the increase of the number of layers means a transition to the space of another, larger dimension. So, it turns, that the $P(z)$ function also contains information on how to decrease the merit function due to an increase of the sought for vector dimension. Let us note that all the standard optimization methods always operate in the space of a fixed dimension.

### 2.2.3 Description of the method

We will consider the simplest version of the method which was suggested by Baskakov and Tikhonravov (1984). It consists of 3 major components: 1-the merit function optimization by the gradient methods; 2-the $P(z)$ function calculation and the needle-like variations block; 3projection of the layer thicknesses vector on spaces of smaller dimension. Let us now look upon each one separately, concentrating on their goals and relations.

The first component is a program of the merit function optimization described in the previous section. The outcome of its work is an $m$-dimension vector, whose coordinates are the layer thicknesses of a two-component coating.

The second component program is the principal one in the method under investigation. It operates as follows. The $m_{1}$-dimension vector $X_{1}$ describing the layer thicknesses of some coating is fed at the input of the program. It can either be a vector obtained as a result of optimization with the preset starting design $X_{0}$, or the vector selected from the series of the local minima for the randomly set starting designs (see Section 2.1 ), or some arbitrary vector. The $P(z)$ function corresponding to the $n(z)$ distribution described by vector $X_{1}$ is calculated here. The $P(z)$ calculation algorithm will be considered in details below after we describe and investigate the method as a whole. Then a point is determined where the needle-like variation leads to a most effective decrease of the merit function (see above). At this point a permissible needle-like variation of the width that actually decreases the merit function is effected (at large values of $\Delta z$ there may be no decrease, as was discussed earlier). For the simplest case, the $\Delta z$ selection is effected in the following way. A certain initial width of the needle (about 10 nm for
the synthesis in the visible band of the spectrum) is set as an input parameter of the method. If it turns to be too large, a successive decrease of $\Delta z$ occurs to ensure the needle-like variation bringing about the merit function decrease. Then, the $P(z)$ function is calculated again and the most favorable point of variation is determined and another permissible needle-like variation is effected resulting in the new decrease of the merit function. This process is reiterated $M$ times, where $M$ is an input parameter of the method set at the beginning of the calculations (parameter $M$ is usually taken as equal to 3-5).

As a result of a series of needle-like variations, at the output of the second component program $X_{2}$ vector is obtained of the $m_{2} \geq m_{1}$ dimension (the dimension may not necessarily increase, if all the variations were effected at the boundary points). An obligatory condition $F\left(X_{2}\right)<F\left(X_{1}\right)$ is observed.

The vector $X_{2}$ is fed as the starting design for optimization into the first program. The program effects the optimization, the non-negative layer thicknesses taken into account. Many coordinates of the $X_{2}$ vector are small. So, as a rule a number of coordinates of the vector obtained in the process of optimization become equal to zero. Let us denote the vector obtained as $X_{3}$ and its dimension as $m_{3}\left(m_{3} \leq m_{2}\right)$. The value of the merit function further decreases in the process of optimization: $F\left(X_{3}\right)<F\left(X_{2}\right)$.

One of the major obstacles for a good implementability of the results of the synthesis can reveal itself in a great number of the coating layers. Despite the fact that $m_{3}<m_{2}$ as a rule, the $X_{3}$ vector can still be of a too large dimension. So, it turned out to be worthwhile to introduce another component of the method decreasing the dimension. It will simply operate proceeding from the fact that all the coordinates of vector $X_{3}$ having values smaller than certain $d_{\text {min }}$ (which is another input parameter of the method) are believed to be equal to zero. If the coordinate of the inner layer of the coating is taken as equal to zero, the adjacent layers merge into one and the vector dimension decreases by 2 . As a result, vector $X_{4}$ is obtained at the output of the third component, the vector dimensions being $m_{4} \leq m_{3}$. The value of $F\left(X_{4}\right)$ is most likely greater than $F\left(X_{3}\right)$. However, this increase of the merit function can be compensated at least partly by the successive optimization with the first component program. After such an
optimization we obtain vector $X_{5}$ where $F\left(X_{5}\right)<F\left(X_{4}\right)$. The dimension of the $X_{5}$ vector can further decrease due to the limitations taken account of.

The complete cycle of the method terminates on obtaining vector $X_{5}$. Then vector $X_{5}$ can either be fed into the input of the second program to search for the solution according to the above pattern. It may also be presented as the final result of the synthesis. It is very often sufficient for a good result to complete one cycle. The supplied description makes it clear that the method is especially worthwhile in an interactive regime with a possibility of an intellectual impact at every stage (correcting the parameters of the method, solving the question concerning the necessity of a further search, etc.). Note also that one complete cycle of the method takes as a rule a few minutes of calculations on computers.

Before we consider an example of synthesis, let us write down the algorithm of the $P(z)$ function calculation being the basis of the method. According to the above considerations, first we ought to solve the differential equation for admittance $L$ times with $\lambda=\lambda_{1}, \ldots, \lambda_{L}$. Note, that we are now interested not only in the input admittance, necessary to calculate the reflectance, but to ensure the whole of the solution of equation $(2.2 .2)$ on $\left[0, z_{a}\right]$, since it is incorporated in the conjugate equation (2.2.14) as a coefficient. This solution was in fact obtained for the case of a piecewise-constant dependence of $n(z)$ in Section 1.2. Making use of (1.2.20) we can write it down for any layer of the coating. In the $j$-th layer, i.e., on the $\left[z_{j-1}, z_{j}\right]$ segment admittance looks like

$$
\begin{align*}
& A(z, \lambda)=\frac{i n_{j} \sin \varphi+A\left(z_{j-1}, \lambda\right) \cos \varphi}{\cos \varphi+\left(i / n_{j}\right) A\left(z_{j-1}, \lambda\right) \sin \varphi},  \tag{2.2.20}\\
& \varphi=\frac{2 \pi}{\lambda} n_{j}\left(z-z_{j-1}\right) .
\end{align*}
$$

Here $A\left(z_{j-1}, \lambda\right)$ is the value of admittance at the left boundary of the layer. Refractive index $n_{j}$ takes the values of $n_{H}$ and $n_{L}$ depending on the order of sequence of layers with high and low refractive indices. Solution of equation (2.2.2) is obtained with the help of (2.2.20) subsequently from layer to layer, taking account of the initial condition at $z_{0}=0$ :

$$
A\left(z_{0}, \lambda\right)=n_{s} .
$$

Now we will obtain a common formula for the solution of a conjugate equation (2.2.14) in the $j$-th layer. It is necessary for the purpose to substitute the explicit expression for $A(z, \lambda)$
into (2.2.14). Note that if we additionally multiply the numerator in (2.2.20) by $i(2 \pi / \lambda)$ it will coincide with the denominator derivative of the expression. So equation (2.2.14) can be written down in the following form:

$$
\frac{d \Psi}{d z}=2 \frac{d}{d z} \ln \left\{\cos \left[\frac{2 \pi}{\lambda} n_{j}\left(z-z_{j-1}\right)\right]+i \frac{A\left(z_{j-1}, \lambda\right)}{n_{j}} \sin \left[\frac{2 \pi}{\lambda} n_{j}\left(z-z_{j-1}\right)\right]\right\} \Psi(z, \lambda)
$$

Dividing it by the $\Psi(z, \lambda)$ ) function and integrating with respect to $d z$, we obtain a common expression for the conjugate function in the $j$-th layer:

$$
\begin{equation*}
\Psi(z, \lambda)=C\left\{\cos \left[\frac{2 \pi}{\lambda} n_{j}\left(z-z_{j-1}\right)\right]+i \frac{A\left(z_{j-1}, \lambda\right)}{n_{j}} \sin \left[\frac{2 \pi}{\lambda} n_{j}\left(z-z_{j-1}\right)\right]\right\}^{2} . \tag{2.2.21}
\end{equation*}
$$

The boundary conditions $(2.2 .15)$ for the conjugate function are set at the right boundary of the last, $m$-th layer $z_{m}=z_{a}$. So the solution of the conjugate equation ought to be effected from the right leftward. Due to this, the indefinite constant $C$ in (2.2.21) ought to be determined at the right boundary point of the $j$-th layer. As a result we obtain the following expression for calculating the conjugate function in the $j$-th layer

$$
\begin{equation*}
\Psi(z, \lambda)=\Psi\left(z_{j}, \lambda\right)\left\{\frac{\cos \varphi+\frac{i}{n_{j}} A\left(z_{j-1}, \lambda\right) \sin \varphi}{\cos \left(\frac{2 \pi}{\lambda} n_{j} d_{j}\right)+\frac{i}{n_{j}} A\left(z_{j-1}, \lambda\right) \sin \left(\frac{2 \pi}{\lambda} n_{j} d_{j}\right)}\right\}^{2} . \tag{2.2.22}
\end{equation*}
$$

Here $\varphi=\frac{2 \pi}{\lambda} n_{j}\left(z-z_{j-1}\right)$ and $d_{j}$ is the thickness of the j -th layer.


Figure 2.5: Reflectance of the 8 -layer antireflection coating:

$$
\begin{aligned}
& 8 \mathrm{~S} .295 \mathrm{H} .291 \mathrm{~L} .149 \mathrm{H} .074 \mathrm{~L} 1.915 \mathrm{H} .107 \mathrm{~L} .311 \mathrm{H} 1.052 \mathrm{~L} \\
& \quad \text { with } n_{s}=1.52, n_{H}=2.30, n_{L}=1.38 \quad\left(\lambda_{0}=500 \mathrm{~nm}\right)
\end{aligned}
$$

With the help of (2.2.22) for $\lambda=\lambda_{1}, \ldots, \lambda_{L}$ from layer to layer the $\Psi\left(z, \lambda_{l}\right)$ function is calculated. This done, there are no obstacles to finding $P(z)$ from (2.2.18). Note, that it is sufficient to store in the computer memory only the values of the admittance at the layers boundaries in order to determine the $\Psi\left(z, \lambda_{l}\right)$ functions. As we can see, the algorithm of the $P(z)$ function calculation is very economical both with respect to the calculation time and the volume of information stored in the computer memory.

Let us now consider an example of synthesis described by Baskakov and Tikhonravov (1984). An antireflection coating was designed for the visible spectral band on the basis of materials with the refractive indices of $n_{H}=2.30, n_{L}=1.38$ with glass featuring $n_{s}=1.52$ as the substrate and air as the outer space. The layer adjacent to the substrate had a high refractive index.

The synthesis process started with a search for a two-layer antireflection coating with a random setting of the starting design. The time of the search was limited to 5 minutes. The $X_{1}$ vector selected within this time and implementing the deepest local minimum was fed at the input of the second program. Then the synthesis process continued along the pattern described above. The calculations were made within one complete cycle of the method. The reflectance of the synthesized coating is provided in Fig 2.5. The geometric thicknesses of the layers in
nanometers (starting from the substrate) are equal to $14.1,26.4,8.1,6.7,104.1,9.7,16.9,95.3$. The optical thicknesses of the layers are equal to $32.4,36.4,18.6,9.3,239.4,13.4,38.9,131.5$. The total search time was about 10 minutes on the BESM- 6 mainframe.

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[^0]:    ${ }^{1}$ This selection of the time factor gives the sign of the imaginary part of the complex refractive index most frequently used in multilayer optics; it will feature the negative sign. As is seen from below, the change of the sign before $i \omega t$ is equivalent to the substitute of the complex values obtained below for the complex conjugates. Note, that in a number of studies, including the widely known book "Principles of Optics" by M. Born and E. Wolf, the exponential factor takes the form of exp( $-i \omega t$ ) ). So the complex refractive index in that book has a positive imaginary part.

[^1]:    ${ }^{2}$ The selection of the sign preceding $i \vec{k} \vec{r}$ is determined by the selection of the sign before $i \omega t$. The plane of the constant phase $i \vec{k} \vec{r}-\omega t=$ const spreads with time in the direction of the vector $\vec{k}$.

[^2]:    ${ }^{3}$ In mathematics and mechanics, the trajectories of differential equations and space where these trajectories are plotted, are customarily known as phase trajectories and space. It will be only natural to include the problems under consideration here into those covered by the term.

